One-dimensional random field Kac's model: weak large deviations principle

Enza Orlandi, Pierre Picco

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Enza Orlandi [1] and Pierre Picco [2]

Dedicated to A.V. Skorohod for the fiftieth birthday of its fundamental paper [37].

Abstract We prove a quenched weak large deviations principle for the Gibbs measures of a Random Field Kac Model (RFKM) in one dimension. The external random magnetic field is given by symmetrically distributed Bernoulli random variables. The results are valid for values of the temperature, β^{-1}, and magnitude, θ, of the field in the region where the free energy of the corresponding random Curie Weiss model has only two absolute minima m_β and Tm_β. We give an explicit representation of the rate functional which is a positive random functional determined by two distinct contributions. One is related to the free energy cost $\mathcal{F}^*$ to undergo a phase change (the surface tension). The $\mathcal{F}^*$ is the cost of one single phase change and depends on the temperature and magnitude of the field. The other is a bulk contribution due to the presence of the random magnetic field. We characterize the minimizers of this random functional. We show that they are step functions taking values m_β and Tm_β. The points of discontinuity are described by a stationary renewal process related to the $h-$extrema for a bilateral Brownian motion studied by Neveu and Pitman, where h in our context is a suitable constant depending on the temperature and on magnitude of the random field. As an outcome we have a complete characterization of the typical profiles of RFKM (the ground states) which was initiated in [14] and extended in [16].

1 Introduction

We consider a one-dimensional spin system interacting via a ferromagnetic two-body Kac potential and external random magnetic field given by symmetrically distributed Bernoulli random variables. Problems where a stochastic contribution is added to the energy of the system arise naturally in condensed matter physics where the presence of the impurities causes the microscopic structure to vary from point to point. Some of the vast literature on these topics may be found consulting [1-4], [6], [8], [12], [21- 24], [28], [36].

Kac's potentials is a short way to denote two-body ferromagnetic interactions with range $\frac{1}{\gamma}$, where γ is a dimensionless parameter such that when $\gamma \downarrow 0$, i.e. very long range, the strength of the interaction becomes very weak keeping the total interaction between one spin and all the others finite. They were introduced in [25], and then generalized in [29] and [33] to present a rigorous validity of the van der Waals theory of a liquid-vapor phase transition. Performing first the thermodynamic limit of the spin system interacting via Kac's potential, and then the limit of infinite range, $\gamma \downarrow 0$, Lebowitz and Penrose rigorously derived the Maxwell rule, *i.e* the canonical free energy of the system is the convex envelope of the corresponding canonical free energy for the Curie–Weiss model. The consequence is that, in any dimension, for values of the temperature at which the free energy corresponding to the Curie-Weiss model is not convex, the canonical

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free energy of the Kac's model is not differentiable in the limit $\gamma \downarrow 0$. These results show that long range models give satisfactory answer for canonical free energies. At the level of Gibbs measures the analysis is more delicate since the behavior of Gibbs measures depends strongly on the dimension.

There are several papers trying to understand qualitatively and quantitatively how a refined analysis of the Gibbs measures of the Kac models allows to see some features of systems with long, but finite range interaction, see for instance [18], [30], [11].

For γ fixed and different from zero, if $d = 1$, there exists an unique Gibbs state for the Kac model while for the Curie–Weiss model the measure induced by the empirical magnetization weakly converges, when the number of sites goes to infinity, to a convex contribution of two different Dirac measures. In the one dimensional case, the analysis [17] for Ising spin and [9] for more general spin, gives a satisfactory description of the typical profiles. In these papers a large deviations principle for Gibbs measures was established. The ground state of the system in suitable chosen mesoscopic scales, is concentrated sharply near the two values of the minimizers of the corresponding Curie-Weiss canonical free energy. The typical magnetization profiles are constant near one of the two values over lengths of the order $e^{\frac{\beta}{\gamma}F}$ where F was explicitly computed and represents the cost in term of canonical free energy to go from one phase to the other, i.e the surface tension. Moreover, suitably marking the locations of the phase changes of the typical profiles and scaling the space by $e^{-\frac{\beta}{\gamma}F}$, one gets as limiting Gibbs distribution of the marks, the one of a Poisson Point Process. The thermal fluctuations are responsible for the stochastic behavior on this scale.

The same type of questions could be asked for the RFKM which is one of the simplest disordered spin system. This motivated the [14], [16] as well as the present paper. The answers we found, as explained below, are dramatically different from the ones obtained without the presence of the random field. The analysis done holds in dimension $d = 1$ and for values of the temperature and magnitude of the field in the whole region of two absolute minima for the canonical free energy of the corresponding Random Field Curie Weiss model. This region is denoted $\mathcal{E}$, see (2.19) for the precise definition. In the first paper [14] we gave the results for (β, θ) in a subset of $\mathcal{E}$, under some smallness condition, whereas in [16] as well as in this paper we give the result for (β, θ) in $\mathcal{E}$ without further constraints. We will comment later about this, but one should bear in mind that the results proven in [14] hold for almost all realizations of the random magnetic fields, the ones proven in [16] hold for a set of realizations of the random magnetic fields of probability that goes to one when $\gamma \downarrow 0$, while the ones in the present paper hold merely in law.

Let us recall the previous results: Here, as well in the previous papers, the first step is a coarse graining procedure. Through a block-spin transformation, the microscopic system is mapped into a system on $\mathcal{T} = L^\infty(I\!R, [-1, 1]) \times L^\infty(I\!R, [-1, 1])$, see (2.14), for which the length of interaction becomes of order one (the macroscopic system). The macroscopic state of the system is determined by an order parameter which specifies the phase of the system. It has been proven in [14] that for almost all realizations of the random magnetic fields, for intervals whose length in macroscopic scale is of order $(\gamma \log\log(1/\gamma))^{-1}$ the typical block spin profile is either rigid, taking one of the two values (m_β or Tm_β) corresponding to the minima of the canonical free energy of the random field Curie Weiss model, or makes at most one transition from one of the minima to the other. In the following, we will denote these two minima the $+$ or $-$ phases. It was also proven in [14], that if the system is considered on an interval of length $\frac{1}{\gamma}(\log \frac{1}{\gamma})^p$, $p \geq 2$, the typical profiles are not rigid over any interval of length larger or equal to $L_1(\gamma) = \frac{1}{\gamma}(\log \frac{1}{\gamma})(\log\log \frac{1}{\gamma})^{2+\rho}$, for any $\rho > 0$.

In [16] the following was proved: On a set of realizations of the random field of overwhelming probability (when $\gamma \to 0$) it is possible to construct random intervals of length of order $\frac{1}{\gamma}$ (macro scale) and to associate a random sign in such a way that, typically with respect to the Gibbs measure, the magnetization profile is rigid on these intervals and, according to the sign, it belongs to the $+$ or $-$ phase. Hereafter, "random" means that it depends on the realizations of the random fields (and on β, θ). A description of the transition from one phase to the other was also discussed in [16]. We recall these results in Section 2. The main problem in the proof of the previous results is the "non locality" of the system, due to the presence of the random field. There is an interplay between the ferromagnetic two-body interaction which attracts spins alike and the presence of the random field which would like to have the spins aligned according to its sign. It is relatively easy to see that the fluctuations of the random field over intervals in macro scale $\frac{1}{\gamma}$ play an important role. To determine the beginning and the end of the random interval where the profiles are rigid and the sign attributed to it, it is essential to verify other local requirements for the random field. We need a detailed analysis of suitable functions of the random fields in all subintervals of the interval of order $\frac{1}{\gamma}$. In fact, it could happen that even though at large the random fields undergoe to a positive (for example) fluctuation, locally there are negative fluctuations which make not convenient (in terms of the cost of the total free energy) for the system to have a magnetization profile close to the + phase in that interval.

Another problem in the previous analysis is due to the fact that the measure induced by the block-spin transformation contains multibody interaction of arbitrary order. Estimated roughly as in [14], this would give a contribution proportional to the length of the interval in which the transformation is done, there the length of intervals was $(\gamma \log\log(1/\gamma))^{-1}$ and the $(\log\log(1/\gamma))^{-1}$ help us to get a small contribution. Here we are interested in intervals of length $\frac{1}{\gamma}$. Luckily enough, exploiting the randomness of the one body interaction, it is enough to estimate the Lipschitz norm of the multibody potential. Using cluster expansion tools, this can be estimated through the representation of the multibody interaction as an absolute convergent series.

In this paper we first extend the results of [16] by defining a random profile u^*_γ which belongs to $BV([-Q(\gamma), Q(\gamma)], \{m_\beta, Tm_\beta\})$, the set of function from $[-Q(\gamma), Q(\gamma)]$ to $\{m_\beta, Tm_\beta\}$ having bounded variation. Here $Q(\gamma) \uparrow \infty$ when $\gamma \downarrow 0$ in a convenient way. On a probability subspace of the random magnetic field configurations of overwhelming probability, we identify a suitable neighborhood of u^*_γ that has a overwhelming Gibbs measure.

Then we prove that when $\gamma \downarrow 0$ the limiting distribution of the interdistance between the jump points of u^*_γ with respect to the distribution of the random magnetic fields is the Neveu-Pitman [32] stationary renewal process of h-extrema of a bilateral Brownian motion. The value of h depends on β and θ. Surprisingly the residual life distribution of the renewal process that we obtained is the same (setting $h = 1$) of the one determined independently by Kesten [26] and Golosov [22] representing the limit distribution of the point of localization of Sinai's random walk in random environment, see Remark 2.7, in Section 2.

This allows us to define the limiting (in Law) typical profile u^* that belongs to $BV_{\rm loc}(I\!\!R, \{m_\beta, Tm_\beta\})$, the set of functions from $I\!\!R$ to $\{m_\beta, Tm_\beta\}$ that have bounded variations on each finite interval of $I\!\!R$. The total variation of u^* on $I\!\!R$ is infinite.

Note that here, the Gibbs measure is strongly concentrated on a random profile that we relate to a renewal process, the randomness being the one of the random magnetic fields. The phase change of this random profile occurs on such a small scale that we cannot see the thermal fluctuations that were responsible in the case without magnetic field of the previously described Poisson Point Process. At the same scale where we find the renewal process, the system without magnetic fields is completely rigid, constantly equal to m_β or Tm_β. Having exhibited the typical profile u^*_γ and its limit in Law u^* the next natural question concerns the large deviations with respect to this typical profile. Formally we would like to determine a positive functional $\Gamma(u)$ for $u \in \mathcal{A}$, where $\mathcal{A} \subset \mathcal{T}$, so that

$$\mu^\omega_\gamma[\mathcal{A}] \sim \exp\{-\frac{\beta}{\gamma} \inf_{u \in \mathcal{A}} \Gamma(u)\}. \tag{1.1}$$

When $\mathcal{A} \equiv \mathcal{A}(u) \subset \mathcal{T}$ is a convenient, see (2.33), neighborhood of $u \in BV_{\rm loc}(I\!\!R, \{Tm_\beta, m_\beta\})$ and u is a suitable local perturbation of the typical profile u^*_γ, (1.1) should be understood as

$$\lim_{\gamma\downarrow 0}\left[-\frac{\gamma}{\beta}\log\mu^\omega_\gamma[\mathcal{A}(u)]\right]=\Gamma(u). \tag{1.2}$$

One has to give a probabilistic sense to the above convergence. It appears that contrarily to the large deviation functional associated to the global empirical magnetization (the canonical free energy of the RFKM, see (2.18)) which is not random, $\Gamma(u)$ is random and the above convergence holds in Law. In fact $\Gamma(u)$ can be expressed in term of u, the limiting u^* and the bilateral Brownian motion. It represents, in the chosen limit, the random cost for the system to deviate from the equilibrium value u^*. The interplay between the surface free energy $\mathcal{F}^*$ (the cost of one single phase change) and the random bulk contribution appears in a rather clear way. Note that in (1.2) the functional is evaluated at u even if the considered neighborhood of u does not shrink when $\gamma\downarrow 0$ to u. This fact allows us to avoid to face difficult measurability problems when performing infimum over family of sets. The random functional Γ in (1.2) could be seen as the " De Giorgi Gamma-limit in Law" for a sequence of intermediate random functionals obtained through a coarse graining procedure over $\mathcal{T}$. Since a precise definition of such a convergence is beyond the scope of the paper, see however [19], and presents more complications than simplifications we will not pursue it here.

The plan of the paper is the following. In Section 2 we give the description of the model and present the main results. In Section 3 we recall the coarse graining procedure. In section 4 we prove the main estimates to derive upper and lower bound to deduce the large deviation estimates. In Section 5 we prove the above mentioned convergence in Law of the localization of the jumps of u^*_γ to the stationary renewal process of Neveu–Pitman. In section 6 we give the proof of the main results.

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2 Model, notations and main results

2.1. The model

Let $(\Omega,\mathcal{A},I\!\!P)$ be a probability space on which we define $h\equiv\{h_i\}_{i\in\mathbb{Z}}$, a family of independent, identically distributed Bernoulli random variables with $I\!\!P[h_i=+1]=I\!\!P[h_i=-1]=1/2$. They represent random signs of external magnetic fields acting on a spin system on $\mathbb{Z}$, and whose magnitude is denoted by $\theta>0$. The configuration space is $\mathcal{S}\equiv\{-1,+1\}^{\mathbb{Z}}$. If $\sigma\in\mathcal{S}$ and $i\in\mathbb{Z}$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by a Kac potential of the form $J_\gamma(i-j)\equiv\gamma J(\gamma(i-j))$, $\gamma>0$. We require that for $r\in I\!\!R$: (i) $J(r)\ge 0$ (ferromagnetism); (ii) $J(r)=J(-r)$ (symmetry); (iii) $J(r)\le ce^{-c'|r|}$ for c,c' positive constants (exponential decay); (iv) $\int J(r)dr=1$ (normalization). For sake of simplicity we fix $J(r)=1\!\!1_{[|r|\le 1/2]}(r)$, where we denote by $1\!\!1_A(\cdot)$ the indicator function of the set A.

For $\Lambda\subseteq\mathbb{Z}$ we set $\mathcal{S}_\Lambda=\{-1,+1\}^\Lambda$; its elements are denoted by σ_Λ; also, if $\sigma\in\mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda\subset\mathbb{Z}$ finite and a realization of the magnetic fields, the Hamiltonian in the volume Λ, with free boundary conditions, is the random variable on $(\Omega,\mathcal{A},I\!\!P)$ given by

$$H_\gamma(\sigma_\Lambda)[\omega]=-\frac{1}{2}\sum_{(i,j)\in\Lambda\times\Lambda}J_\gamma(i-j)\sigma_i\sigma_j-\theta\sum_{i\in\Lambda}h_i[\omega]\sigma_i. \tag{2.1}$$

In the following we drop the ω from the notation.

The corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$ and free boundary condition is then a random variable with values on the space of probability measures on $\mathcal{S}_\Lambda$. We denote it by $\mu_{\beta,\theta,\gamma,\Lambda}$ and it is defined by

$$\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \exp\{-\beta H_\gamma(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.2}$$

where $Z_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor called partition function. To take into account the interaction between the spins in Λ and those outside Λ we set

$$W_\gamma(\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J_\gamma(i-j)\sigma_i\sigma_j. \tag{2.3}$$

If $\tilde\sigma \in \mathcal{S}$, the Gibbs measure on the finite volume Λ and boundary condition $\tilde\sigma_{\Lambda^c}$ is the random probability measure on $\mathcal{S}_\Lambda$, denoted by $\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ and defined by

$$\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}} \exp\left\{-\beta(H_\gamma(\sigma_\Lambda) + W_\gamma(\sigma_\Lambda, \tilde\sigma_{\Lambda^c}))\right\}, \tag{2.4}$$

where again the partition function $Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor.

Given a realization of h and $\gamma > 0$, there is a unique weak-limit of $\mu_{\beta,\theta,\gamma,\Lambda}$ along a family of volumes $\Lambda_L = [-L, L] \cap \mathbb{Z}$, $L \in \mathbb{N}$; such limit is called the infinite volume Gibbs measure $\mu_{\beta,\theta,\gamma}$. The limit does not depend on the boundary conditions, which may be taken h-dependent, but it is a random element, i.e., different realizations of h give a priori different infinite volume Gibbs measures.

2.2. Scales

When dealing with local long range interaction, as we did in [17], [14] and [16], the analysis of the configurations that are typical for $\mu_{\beta,\theta,\gamma}$ in the limit $\gamma \downarrow 0$, involves a block spin transformation which transforms the microscopic system on $\mathbb{Z}$ in a system on $\mathbb{R}$. Such changes of scales are standard in Kac type problems. Here, notations are particularly troublesome because we have three main different scales and according to the case it is better to work with one or the other. There will be also intermediate scales that we will discuss later. For historical reasons the three main scales are called: microscopic, macroscopic and Brownian scale. More properly they should be denoted microscopic, mesoscopic and macroscopic. Since in the previous papers, [14] and [16], the intermediate scale was called *macroscopic*, we continue to call it in such a way to avoid confusion. Then we will call mesoscopic scales all the intermediate scales between the microscopic and macroscopic scales. These mesoscopic scales are not intrinsic to the system but superimposed to study it.

- *The microscopic and macroscopic scales.*

The basic space is the "microscopic space", i.e. the lattice $\mathbb{Z}$ whose elements are denoted by i, j and so on. The microscopic scale corresponds to the length measured according to the lattice distance. The spin σ_i are indexed by $\mathbb{Z}$ and the range of interaction in this scale is of order $\frac{1}{\gamma}$.

The macroscopic regions correspond to intervals of $\mathbb{R}$ that are of order $\frac{1}{\gamma}$ in the microscopic scale ; *i.e.* if $I \subset \mathbb{R}$, is an interval in the macroscopic scale then it will correspond to the interval $\frac{I}{\gamma}$ in the microscopic scale. Since the range of the interaction is of order γ^{-1} in the microscopic scale, in the macroscopic scale it becomes of order 1.

- *The Brownian scale*

The Brownian scale is linked to the random magnetic fields. The Brownian regions correspond to intervals of $I\!R$ that are of order $\frac{1}{\gamma^2}$ in the microscopic scale; i.e. if $[-Q, Q] \subset I\!R$, $Q > 0$ is an interval in Brownian scale then it will correspond to $[-\frac{Q}{\gamma^2}, \frac{Q}{\gamma^2}]$ in the microscopic scale. In the Brownian scale the range of interaction is of order γ.

- *The partition of $I\!R$.*

Given a rational positive number δ, $\mathcal{D}_\delta$ denotes the partition of $I\!R$ into intervals $\tilde{A}_\delta(u) = [u\delta, (u+1)\delta)$ for $u \in Z\!\!Z$. If $\delta = n\delta'$ for some $n \in I\!N$, then $\mathcal{D}_\delta$ is coarser than $\mathcal{D}_{\delta'}$. A function $f(\cdot)$ on $I\!R$ is $\mathcal{D}_\delta$–measurable if it is constant on each interval of $\mathcal{D}_\delta$. A region Λ is $\mathcal{D}_\delta$–measurable if its indicator function is $\mathcal{D}_\delta$–measurable. For $r \in I\!R$, we denote by $D_\delta(r)$ the interval of $\mathcal{D}_\delta$ that contains r. Note that for any $r \in [u\delta, (u+1)\delta)$, we have that $D_\delta(r) = \tilde{A}_\delta(u)$. To avoid rounding problems in the following, we will consider intervals that are always $\mathcal{D}_\delta$–measurable. If $I \subseteq I\!R$ denotes a macroscopic interval we set

$$\mathcal{C}_\delta(I) = \{u \in Z\!\!Z; \tilde{A}_\delta(u) \subseteq I\}. \tag{2.5}$$

- *The mesoscopic scales*

The smallest mesoscopic scale involves a parameter $0 < \delta^*(\gamma) < 1$ satisfying certain conditions of smallness that will be fixed later. However we assume that $\delta^*\gamma^{-1} \uparrow \infty$ when $\gamma \downarrow 0$. The elements of $\mathcal{D}_{\delta^*}$ will be denoted by $\tilde{A}(x) \equiv [x\delta^*, (x+1)\delta^*)$, with $x \in Z\!\!Z$. The partition $\mathcal{D}_{\delta^*}$ induce a partition of $Z\!\!Z$ into blocks $A(x) = \{i \in Z\!\!Z; i\gamma \in \tilde{A}(x)\} \equiv \{a(x), \ldots, a(x+1) - 1\}$ with length of order $\delta^*\gamma^{-1}$ in the microscopic scale.

For notational simplicity, if no confusion arises, we omit to write the explicit dependence on γ, δ^*. To avoid rounding problems, we assume that $\gamma = 2^{-n}$ for some integer n, with δ^* such that $\delta^*\gamma^{-1}$ is an integer, so that $a(x) = x\delta^*\gamma^{-1}$, with $x \in Z\!\!Z$. When considering another mesoscopic scale, say $\delta > \delta^*$, we always assume that $\delta^{-1} \in I\!N$ and $\delta = k\delta^*$ for some integer $k \geq 2$.

2.3 Basic Notations.

- *block-spin magnetization*

Given a realization of h and for each configuration σ_Λ, we could have defined for each block $A(x)$ a pair of numbers where the first is the average magnetization over the sites with positive h and the second to those with negative h. However it appears, [14], to be more convenient to use another random partition of $A(x)$ into two sets of the same cardinality. This allows to separate on each block the expected contribution of the random field from its local fluctuations. More precisely we have the following.

Given a realization $h[\omega] \equiv (h_i[\omega])_{i \in Z\!\!Z}$, we set $A^+(x) = \{i \in A(x); h_i[\omega] = +1\}$ and $A^-(x) = \{i \in A(x); h_i[\omega] = -1\}$. Let $\lambda(x) \equiv \mathrm{sgn}(|A^+(x)| - (2\gamma)^{-1}\delta^*)$, where sgn is the sign function, with the convention that $\mathrm{sgn}(0) = 0$. For convenience we assume $\delta^*\gamma^{-1}$ to be even, in which case:

$$I\!P\left[\lambda(x) = 0\right] = 2^{-\delta^*\gamma^{-1}} \binom{\delta^*\gamma^{-1}}{\delta^*\gamma^{-1}/2}. \tag{2.6}$$

We note that $\lambda(x)$ is a symmetric random variable. When $\lambda(x) = \pm 1$ we set

$$l(x) \equiv \inf\{l \geq a(x) \,:\, \sum_{j=a(x)}^{l} 1\!\!1_{\{A^{\lambda(x)}(x)\}}(j) \geq \delta^*\gamma^{-1}/2\} \tag{2.7}$$

and consider the following decomposition of $A(x)$: $B^{\lambda(x)}(x) = \left\{i \in A^{\lambda(x)}(x); i \le l(x)\right\}$ and $B^{-\lambda(x)}(x) = A(x) \setminus B^{\lambda(x)}(x)$. When $\lambda(x) = 0$ we set $B^+(x) = A^+(x)$ and $B^-(x) = A^-(x)$. We set $D(x) \equiv A^{\lambda(x)}(x) \setminus B^{\lambda(x)}(x)$. In this way, the set $B^{\pm}(x)$ depends on the realizations of the random field, but the cardinality $|B^{\pm}(x)| = \delta^*\gamma^{-1}/2$ is the same for all realizations. Set

$$m^{\delta^*}(\pm, x, \sigma) = \frac{2\gamma}{\delta^*} \sum_{i \in B^{\pm}(x)} \sigma_i. \tag{2.8}$$

We call block spin magnetization of the block $A(x)$ the vector

$$m^{\delta^*}(x, \sigma) = (m^{\delta^*}(+, x, \sigma), m^{\delta^*}(-, x, \sigma)). \tag{2.9}$$

The total empirical magnetization of the block $A(x)$ is, of course, given by

$$\frac{\gamma}{\delta^*} \sum_{i \in A(x)} \sigma_i = \frac{1}{2}(m^{\delta^*}(+, x, \sigma) + m^{\delta^*}(-, x, \sigma)) \tag{2.10}$$

and the contribution of the magnetic field to the Hamiltonian (2.1) is

$$\frac{\gamma}{\delta^*} \sum_{i \in A(x)} h_i \sigma_i = \frac{1}{2}(m^{\delta^*}(+, x, \sigma) - m^{\delta^*}(-, x, \sigma)) + \lambda(x) \frac{2\gamma}{\delta^*} \sum_{i \in D(x)} \sigma_i. \tag{2.11}$$

- *spaces of the magnetization profiles*

Given a volume $\Lambda \subseteq Z\!\!\!Z$ in the original microscopic spin system, it corresponds to the macroscopic volume $I = \gamma\Lambda = \{\gamma i; i \in \Lambda\}$, assumed to be $\mathcal{D}_{\delta^*}$–measurable. The block spin transformation, as considered in [14] and [16], is the random map which associates to the spin configuration σ_Λ the vector $(m^{\delta^*}(x, \sigma))_{x \in \mathcal{C}_{\delta^*}(I)}$, see (2.9), with values in the set

$$\mathcal{M}_{\delta^*}(I) \equiv \prod_{x \in \mathcal{C}_{\delta^*}(I)} \left\{-1, -1 + \frac{4\gamma}{\delta^*}, -1 + \frac{8\gamma}{\delta^*}, \dots, 1 - \frac{4\gamma}{\delta^*}, 1\right\}^2. \tag{2.12}$$

We use the same notation $\mu_{\beta,\theta,\gamma,\Lambda}$ to denote both, the Gibbs measure on $\mathcal{S}_\Lambda$, and the probability measure induced on $\mathcal{M}_{\delta^*}(I)$, through the block spin transformation, i.e., a coarse grained version of the original measure. Analogously, the infinite volume limit (as $\Lambda \uparrow Z\!\!\!Z$) of the laws of the block spin $(m^{\delta^*}(x))_{x \in \mathcal{C}_{\delta^*}(I)}$ under the Gibbs measure will also be denoted by $\mu_{\beta,\theta,\gamma}$.

We denote a generic element in $\mathcal{M}_{\delta^*}(I)$ by

$$m_I^{\delta^*} \equiv (m^{\delta^*}(x))_{x \in \mathcal{C}_{\delta^*}(I)} \equiv (m_1^{\delta^*}(x), m_2^{\delta^*}(x))_{x \in \mathcal{C}_{\delta^*}(I)}. \tag{2.13}$$

Since I is assumed to be $\mathcal{D}_{\delta^*}$–measurable, we can identify $m_I^{\delta^*}$ with the element of

$$\mathcal{T} = \{m \equiv (m_1, m_2) \in L^\infty(I\!\!R) \times L^\infty(I\!\!R); \|m_1\|_\infty \vee \|m_2\|_\infty \le 1\} \tag{2.14}$$

piecewise constant, equal to $m^{\delta^*}(x)$ on each $\tilde{A}(x) = [x\delta^*, (x+1)\delta^*)$ for $x \in \mathcal{C}_{\delta^*}(I)$, and vanishing outside I. Elements of $\mathcal{T}$ will be called magnetization profiles. Recalling that $I = \gamma\Lambda$, the block spin transformation can be identified with a map from the space of spin configurations $\{-1, +1\}^\Lambda$ (with Λ a microscopic volume) into the subset of $\mathcal{D}_{\delta^*}$–measurable functions of $L^\infty(I) \times L^\infty(I)$ (with $I = \gamma\Lambda$ a macroscopic volume).

For $\delta \geq \delta^*$, recalling that $\forall r \in [u\delta, (u+1)\delta)$, we have $D_\delta(r) = \tilde{A}_\delta(u)$, we define for $m = (m_1, m_2) \in \mathcal{T}$ and $i = 1, 2$

$$m_i^\delta(r) = \frac{1}{\delta} \int_{D_\delta(r)} m_i(s) ds. \tag{2.15}$$

This defines a map from $\mathcal{T}$ into the subset of $\mathcal{D}_\delta$–measurable functions of $\mathcal{T}$. We define also a map from $\mathcal{T}$ into itself by

$$(Tm)(x) = (-m_2(x), -m_1(x)) \quad \forall x \in I\!R. \tag{2.16}$$

In the following we denote the total magnetization at the site $x \in I\!R$

$$\tilde{m}(x) = \frac{m_1(x) + m_2(x)}{2}. \tag{2.17}$$

- *The Random Field Curie–Weiss model*

The Lebowitz -Penrose theory, [29], is easy to prove for the Random Field Kac Model see [14], Theorem 2.2. Namely, performing first the thermodynamic limit of the spin system interacting via Kac's potential and then the limit of infinite range, $\gamma \to 0$, the canonical free energy of the Random Field Kac model is the convex envelope of the corresponding canonical free energy for the Random Field Curie-Weiss model.

The canonical free energy for the Random Field Curie-Weiss model derived in [17] is

$$f_{\beta,\theta}(m_1, m_2) = -\frac{(m_1 + m_2)^2}{8} - \frac{\theta}{2}(m_1 - m_2) + \frac{1}{2\beta}(\mathcal{I}(m_1) + \mathcal{I}(m_2)), \tag{2.18}$$

where $\mathcal{I}(m) = \frac{(1+m)}{2} \log\left(\frac{1+m}{2}\right) + \frac{(1-m)}{2} \log\left(\frac{1-m}{2}\right)$. In Section 9 of [16], it was proved that

$$\mathcal{E} = \begin{cases} 0 < \theta < \theta_{1,c}(\beta), & \text{for } 1 < \beta < \frac{3}{2}; \\ 0 < \theta \leq \theta_{1,c}(\beta) & \text{for} \beta \geq \frac{3}{2}, \end{cases} \tag{2.19}$$

where

$$\theta_{1,c}(\beta) = \frac{1}{\beta} \operatorname{arctanh} (1 - \frac{1}{\beta})^{1/2}, \tag{2.20}$$

is the maximal region of the two parameters (β, θ), whose closure contains $(1, 0)$ in which $f_{\beta,\theta}(\cdot, \cdot)$ has exactly three critical points $m_\beta, 0, Tm_\beta$. The two equal minima correspond to $m_\beta = (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$ and 0 a local maximum. Calling $\tilde{m}_\beta = \frac{m_{\beta,1} + m_{\beta,2}}{2}$, on $\mathcal{E}$ we have

$$\frac{\beta}{2\cosh^2(\beta(\tilde{m}_\beta + \theta)} + \frac{\beta}{2\cosh^2 \beta(\tilde{m}_\beta - \theta)} < 1. \tag{2.21}$$

Moreover, for all $(\beta, \theta) \in \mathcal{E}$, the minima are quadratic and therefore there exists a strictly positive constant $\kappa(\beta, \theta)$ so that for each $m \in [-1, +1]^2$

$$f_{\beta,\theta}(m) - f_{\beta,\theta}(m_\beta) \geq \kappa(\beta, \theta) \min\{\|m - m_\beta\|_1^2, \|m - Tm_\beta\|_1^2\}, \tag{2.22}$$

where $\|\cdot\|_1$ is the ℓ^1 norm in $I\!R^2$.

- *The spatially homogeneous phases*

We introduce the so called "excess free energy functional" $\mathcal{F}(m)$, $m \in \mathcal{T}$:

$$\begin{aligned} \mathcal{F}(m) &= \mathcal{F}(m_1, m_2) \\ &= \frac{1}{4} \int \int J(r-r') \left[\tilde{m}(r) - \tilde{m}(r')\right]^2 dr dr' + \int \left[f_{\beta,\theta}(m_1(r), m_2(r)) - f_{\beta,\theta}(m_{\beta,1}, m_{\beta,2})\right] dr \end{aligned} \tag{2.23}$$

with $f_{\beta,\theta}(m_1, m_2)$ given by (2.18) and $\tilde{m}(r) = (m_1(r) + m_2(r))/2$. The functional $\mathcal{F}$ is well defined and non-negative, although it may take the value $+\infty$. Clearly, the absolute minimum of $\mathcal{F}$ is attained at the functions constantly equal to m_β (or constantly equal to Tm_β), the minimizers of $f_{\beta,\theta}$. These two minimizers of $\mathcal{F}$ are called the spatially homogeneous phases. The functional $\mathcal{F}$ represents the continuum approximation of the deterministic contribution to the free energy of the system (cf. (3.3)) normalized by subtracting $f_{\beta,\theta}(m_\beta)$, the free energy of the homogeneous phases. Notice that $\mathcal{F}$ is invariant under the T-transformation, defined in (2.16).

- *The surface tension*

In analogy to systems in higher dimensions, we denote by surface tension the free energy cost needed by the system to undergo to a phase change. It has been proven in [15] that under the condition $m_1(0)+m_2(0) = 0$, and for $(\beta, \theta) \in \mathcal{E}$, there exists a unique minimizer $\bar{m} = (\bar{m}_1, \bar{m}_2)$, of $\mathcal{F}$ over the set

$$\mathcal{M}_\infty = \{(m_1, m_2) \in \mathcal{T}; \limsup_{r \to -\infty} m_i(r) < 0 < \liminf_{r \to +\infty} m_i(r), i = 1, 2\}. \tag{2.24}$$

Without the condition $m_1(0) + m_2(0) = 0$, there is a continuum of minimizers obtained translating $\bar{m}$. The minimizer $\bar{m}(\cdot)$ is infinitely differentiable and converges exponential fast, as $r \uparrow +\infty$ (resp. $-\infty$) to the limit value m_β, (resp.Tm_β). Since $\mathcal{F}$ is invariant by the T-transformation, see (2.16), interchanging $r \uparrow +\infty$ and $r \downarrow -\infty$ in (2.24), there exists one other family of minimizers obtained translating $T\bar{m}$. We denote by

$$\mathcal{F}^* \equiv \mathcal{F}^*(\beta, \theta) = \mathcal{F}(\bar{m}) = \mathcal{F}(T\bar{m}) > 0, \tag{2.25}$$

the surface tension.

- *how to detect local equilibrium*

As in [14], the description of the profiles is based on the behavior of local averages of $m^{\delta^*}(x)$ over k successive blocks in the block spin representation, where $k \geq 2$ is a positive integer. Let $\delta = k\delta^*$ be such that $1/\delta \in I\!N$. Let $\ell \in Z\!\!Z$, $[\ell, \ell+1)$ be a macroscopic block of length 1, $\mathcal{C}_\delta([\ell, \ell+1))$, as in (2.5), and $\zeta > 0$. We define the block spin variable

$$\eta^{\delta,\zeta}(\ell) = \begin{cases} 1, & \text{if } \forall_{u \in \mathcal{C}_\delta([\ell,\ell+1))} \frac{\delta^*}{\delta} \sum_{x \in \mathcal{C}_{\delta^*}([u\delta,(u+1)\delta))} \|m^{\delta^*}(x,\sigma) - m_\beta\|_1 \leq \zeta; \\ -1, & \text{if } \forall_{u \in \mathcal{C}_\delta([\ell,\ell+1))} \frac{\delta^*}{\delta} \sum_{x \in \mathcal{C}_{\delta^*}([u\delta,(u+1)\delta))} \|m^{\delta^*}(x,\sigma) - Tm_\beta\|_1 \leq \zeta; \\ 0, & \text{otherwise.} \end{cases} \tag{2.26}$$

where for a vector $v = (v_1, v_2)$, $\|v\|_1 = |v_1| + |v_2|$. When $\eta^{\delta,\zeta}(\ell) = 1$, (resp. -1), we say that a spin configuration $\sigma \in \{-1, 1\}^{\frac{1}{\gamma}[\ell,\ell+1)}$ has magnetization close to m_β, (resp. Tm_β), with accuracy (δ, ζ) in $[\ell, \ell+1)$. Note that $\eta^{\delta,\zeta}(\ell) = 1$ (resp -1) is equivalent to

$$\forall y \in [\ell, \ell+1) \qquad \frac{1}{\delta} \int_{D_\delta(y)} dx \|m^{\delta^*}(x,\sigma) - v\|_1 \leq \zeta \tag{2.27}$$

for $v = m_\beta$ (resp. Tm_β), since for any $u \in \mathcal{C}_\delta([\ell, \ell+1))$, for all $y \in [u\delta, (u+1)\delta) \subset [\ell, \ell+1)$, $D^\delta(y) = [u\delta, (u+1)\delta)$. We say that a magnetization profile $m^{\delta^*}(\cdot)$, in a macroscopic interval $I \subseteq I\!R$, is close to the equilibrium phase τ, for $\tau \in \{-1, +1\}$, with accuracy (δ, ζ) when

$$\{\eta^{\delta,\zeta}(\ell) = \tau, \, \forall \ell \in I \cap Z\!\!Z\} \tag{2.28}$$

or equivalently if

$$\forall y \in I \qquad \frac{1}{\delta} \int_{D^\delta(y)} dx \| m^{\delta^*}(x, \sigma) - v\|_1 \le \zeta \tag{2.29}$$

where $v = m_\beta$ if $\tau = +1$ and $v = Tm_\beta$ if $\tau = -1$. In the following the letter ℓ will always indicate an element of $Z\!\!Z$. This will allow to write (2.28) as $\{\eta^{\delta,\zeta}(\ell) = \tau, \, \forall \ell \in I\}$. In (2.29) the interval I is always given in the macro–scale. The definition (2.29) can be used for function v more general than the constant ones. In particular, given $v = (v_1, v_2) \in \mathcal{T}$, $\delta = n\delta^*$ for some positive integer n, $\zeta > 0$, and $[a, b)$ an interval in Brownian scale, we say that a spin configuration $\sigma \in \{-1, 1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})}$ has magnetization profile close to v with accuracy (δ, ζ) in the interval $[a, b)$ if σ belongs to the set

$$\left\{ \sigma \in \{-1, 1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})} : \forall y \in [\frac{a}{\gamma}, \frac{b}{\gamma}) \qquad \frac{1}{\delta} \int_{D^\delta(y)} dx \| m^{\delta^*}(x, \sigma) - v^{\delta^*}(x)\|_1 \le \zeta \right\}. \tag{2.30}$$

In view of the results on the typical configurations obtained in [16] the above notion is too strong. In fact the typical profiles form long runs of length of order γ^{-1} (in the macroscopic scale) of $\eta^{\delta,\zeta}(\cdot) = 1$ that are followed by short runs of $\eta^{\delta,\zeta}(\cdot) = 0$ that are in turn followed by long runs of $\eta^{\delta,\zeta}(\cdot) = -1$. The typical profiles undergo to a phase change within the runs of $\eta^{\delta,\zeta}(\cdot) = 0$. The length of these runs, see Theorem 2.4 in [16], is smaller than $2R_2 = 2R_2(\gamma) \uparrow \infty$ in the macroscopic scale, see (2.66). In the Brownian scale, this length becomes $2\gamma R_2$ and one obtains that $\gamma R_2 \downarrow 0$. So in Brownian scale, when $\gamma \downarrow 0$, the localization of the phase change shrinks to a point : the point of a jump. For small $\gamma > 0$, the results in [16] allow to localize these points within an interval of length $2\rho >> 2\gamma R_2$ centered around well defined points depending on the realizations of the random field. We call ρ the fuzziness and $\rho = \rho(\gamma) \downarrow 0$ in the Brownian scale.

With this in mind, a candidate for the limiting support of $\mu_{\beta,\theta,\gamma}$ when $\gamma \downarrow 0$ is an appropriate neighborhood of functions on $I\!R$, (considered in the Brownian scale), taking two values $m_\beta = (m_{\beta,1}, m_{\beta,2})$ or $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$ that have finite variation. To fix the notations, we recall the standard definitions. Let us define, for any bounded interval $[a, b) \subset I\!R$ (in the Brownian scale) $BV([a, b), \{m_\beta, Tm_\beta\})$ as the set of right continuous bounded variation functions on $[a, b)$ with value in $\{m_\beta, Tm_\beta\}$. Since we consider mainly bounded variation functions with value in $\{m_\beta, Tm_\beta\}$, we write $BV([a, b)) \equiv BV([a, b), \{m_\beta, Tm_\beta\})$. Since any bounded variation function u is the difference of two increasing functions, it has a left limit. We call the jump at r the quantity $Du(r) = u(r) - u(r_-)$ where $u(r_-) = \lim_{s \uparrow r} u(s)$. If r is such that $Du(r) \ne 0$ we call r a point of jump of u, and in such a case $\|Du(r)\|_1 = 4\tilde{m}_\beta$. We denote by $N_{[a,b)}(u)$ the number of jumps of u on $[a, b)$ and by $V_a^b(u)$ the variation of u on $[a, b)$, i.e.

$$V_a^b(u) \equiv \sum_{a \le r < b} \|Du(r)\|_1 = N_{[a,b)}(u) 2[m_{\beta,1} + m_{\beta,2}] = 4\tilde{m}_\beta N_{[a,b)}(u) < \infty. \tag{2.31}$$

Note that $Du(r) \ne 0$ only on points of jump of u and therefore the sum in (2.31) is well defined. We denote by $BV_{\rm loc} \equiv BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$ the set of functions from $I\!R$ with values in $\{m_\beta, Tm_\beta\}$ which restricted to any bounded interval have bounded variation but not necessarily having bounded variation on $I\!R$. If $u \in BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$, then, see (2.17), $\tilde{u} \in BV_{\rm loc}(I\!R, \{\tilde{m}_\beta, -\tilde{m}_\beta\})$ where $\tilde{m}_\beta$ is defined before (2.21).

Since a phase change can be better detected in macro units we state the following definition which corresponds to Definition 2.3 of [16].

Definition 2.1 The macro interfaces *Given an interval $[\ell_1, \ell_2]$ (in macro-scale) and a positive integer $2R_2 \le |\ell_2 - \ell_1|$, we say that a single phase change occurs within $[\ell_1, \ell_2]$ on a length R_2 if there exists $\ell_0 \in (\ell_1 + R_2, \ell_2 - R_2)$ so that $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_1) \in \{-1,+1\}, \forall \ell \in [\ell_1, \ell_0 - R_2]$; $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_2) = -\eta(\ell_1), \forall \ell \in [\ell_0 + R_2, \ell_2]$, and $\{\ell \in [\ell_0 - R_2, \ell_0 + R_2] : \eta^{\delta,\zeta}(\ell) = 0\}$ is a set of consecutive integers. We denote by $\mathcal{W}_1([\ell_1, \ell_2], R_2, \zeta)$ the set of configurations $\eta^{\delta,\zeta}$ with these properties.*

In words, on $\mathcal{W}_1([\ell_1, \ell_2], R_2, \zeta)$, there is an unique run of $\eta^{\delta,\zeta} = 0$, with no more than $2R_2$ elements, inside the interval $[\ell_1, \ell_2]$. To take into account that for the typical profiles the point of jumps are determined with fuzziness ρ, it is convenient to associate to $u \in BV([a, b))$ a partition of the interval $[a, b)$ (in Brownian scale) as follows :

Definition 2.2 Partition associated to BV functions *Given $u \in BV([a, b))$, $\rho > \delta = n\delta^*$, with $8\rho + 8\delta$ smaller than the minimal distance between two points of jumps of u, let $C_i(u)$, $i = 1, .., N_{[a,b)}(u)$, (see (2.31)), be the smallest $\mathcal{D}_\delta$ measurable interval that contains an interval of diameter 2ρ, centered at the $i-th$ jump of u in $[a, b)$. We have $C_i(u) \cap C_j(u) = \emptyset$ for $i \ne j$.*

Let $C(u) = \cup_{i=1}^{N_{[a,b)}(u)} C_i(u)$. We set $B(u) = [a, b) \setminus C(u)$ and write $[a, b) = C(u) \cup B(u)$. We denote by $C_{i,\gamma}(u) = \gamma^{-1} C_i(u)$, $C_\gamma(u) = \gamma^{-1} C(u)$ and $B_\gamma(u) = \gamma^{-1} B(u)$ the elements of the induced partition on the macroscopic scale.

Whenever we deal with functions in $\mathcal{T}$ we will always assume that their argument varies on the macroscopic scale. So $m \in \mathcal{T}$ means that $m(x), x \in I$ where $I \subset I\!R$ is an interval in the macroscopic scale. Whenever we deal with bounded variation functions, if not further specified, we will always assume that their argument varies on the Brownian scale. Therefore $u \in BV([a, b))$ means that $u(r), r \in [a, b)$ and $[a, b)$ is considered in the Brownian scale. This means that in the macroscopic scale we need to write $u(\gamma x)$ for $x \in [\frac{a}{\gamma}, \frac{b}{\gamma})$. For $u \in BV([a, b))$, we define for $x \in [\frac{a}{\gamma}, \frac{b}{\gamma})$ *i.e* in the macroscopic scale,

$$u^{\gamma,\delta^*}(x) = \frac{1}{\delta^*} \int_{D_{\delta^*}(x)} u(\gamma s)\, ds. \tag{2.32}$$

Given $[a, b)$ (in the Brownian scale), u in $BV([a, b))$, $\rho > \delta = n\delta^* > 0$, with $8\rho + 8\delta$ satisfying the condition of Definition 2.2, $\zeta > 0$, we say that a spin configuration $\sigma \in \{-1, 1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})}$ has magnetization profile close to u with accuracy (δ, ζ) and *fuzziness* ρ if $\sigma \in \mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[a,b)}(u)$ where

$$\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[a,b)}(u) =$$

$$\left\{ \sigma \in \{-1, 1\}^{[\frac{a}{\gamma^2}, \frac{b}{\gamma^2})} : \forall y \in B_\gamma(u),\ \frac{1}{\delta} \int_{D^\delta(y)} \| m^{\delta^*}(x, \sigma) - u^{\gamma,\delta^*}(x) \|_1\, dx \le \zeta \right\} \bigcap_{i=1}^{N_{[a,b)}(u)} \mathcal{W}_1([C_{i,\gamma}(u)], R_2, \zeta). \tag{2.33}$$

In (2.33) we consider the spin configurations close with accuracy (δ, ζ) to m_β or Tm_β in $B_\gamma(u)$ according to the value of $u^{\gamma,\delta^*}(\cdot)$. In $C_\gamma(u)$ we require that the spin configurations have only one jump in each interval $C_{i,\gamma}(u)$, $i = 1, ..N$, and are close with accuracy (δ, ζ) to the right and to the left of this interval to the value of u in those intervals of $B_\gamma(u)$ that are adjacent to $C_{i,\gamma}(u)$. With all these definitions in hand we can slightly improve the main results of [16].

Theorem 2.3 [COPV] *Given $(\beta,\theta)\in\mathcal{E}$, see (2.19), there exists $\gamma_0(\beta,\theta)$ so that for $0<\gamma\le\gamma_0(\beta,\theta)$, for $Q=\exp[(\log g(1/\gamma))/\log\log g(1/\gamma)]$, with $g(1/\gamma)$ a suitable positive, increasing function such that $\lim_{x\uparrow\infty} g(x)=+\infty$, $\lim_{x\uparrow\infty} g(x)/x=0$ for suitable values of $\delta>\delta^*>0$, $\rho>0$, $\zeta>0$, $a'>0$, R_2 there exists $\Omega_1\subset\Omega$ with*

$$I\!P[\Omega_1]\ge 1-K(Q)\left(\frac{1}{g(1/\gamma)}\right)^{a'} \tag{2.34}$$

where

$$K(Q)=2+5(V(\beta,\theta)/(\mathcal{F}^*)^2)Q\log[Q^2 g(1/\gamma)], \tag{2.35}$$

$\mathcal{F}^*=\mathcal{F}^*(\beta,\theta)$ is defined in (2.25) and

$$V(\beta,\theta)=\log\frac{1+m_{\beta,2}\tanh(2\beta\theta)}{1-m_{\beta,1}\tanh(2\beta\theta)}. \tag{2.36}$$

For $\omega\in\Omega_1$ we explicitly construct $u^*_\gamma(\omega)\in BV([-Q,Q])$ so that the minimal distance between jumps of u^*_γ within $[-Q,+Q]$ is bounded from below by $8\rho+8\delta$,

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,Q]}(u^*_\gamma(\omega))\right)\ge 1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(1/\gamma)}}, \tag{2.37}$$

and

$$V^{Q}_{-Q}(u^*_\gamma)\le 4\tilde{m}_\beta K(Q). \tag{2.38}$$

The previous Theorem is a direct consequence of Theorem 2.1, Theorem 2.2 and Theorem 2.4 proven in [16], together with Lemma 5.14 that gives the value (2.35). The control of the minimal distance between jumps of u^*_γ is done at the end of Section 5.

To facilitate the reading we did not write explicitly in the statement of Theorem 2.3 the choice done of the parameters $\delta,\delta^*,\zeta,g,R_2$ nor the explicit construction of u^*_γ. We dedicate the entire Subsection 2.5 to recall and motivate the choice of the parameters done in [16] as well as in this paper. The $u^*_\gamma(\omega)$ in Theorem 2.3 is a function in $BV([-Q,Q])$ associated to the sequence of maximal elongations and their sign as determined in [16] Section 5. For the moment it is enough to know that it is possible to determine random points $\alpha^*_i=\alpha^*_i(\gamma,\omega)$ and a random sign ±1 associated to intervals $[\epsilon\alpha^*_i,\epsilon\alpha^*_{i+1})$ in the Brownian scale, where $\epsilon=\epsilon(\gamma)$ has to be suitably chosen. These random intervals are the so called maximal elongations. We denote

$$u^*_\gamma(\omega)(r)\equiv\begin{cases} m_\beta, & r\in[\epsilon\alpha^*_i,\epsilon\alpha^*_{i+1})\quad \text{if the sign of elongation } [\epsilon\alpha^*_i,\epsilon\alpha^*_{i+1}) \text{ is } =+1\\ Tm_\beta, & r\in[\epsilon\alpha^*_i,\epsilon\alpha^*_{i+1})\quad \text{if the sign of elongation } [\epsilon\alpha^*_i,\epsilon\alpha^*_{i+1}) \text{ is } =-1.\end{cases} \tag{2.39}$$

for $i\in\{\kappa^*(-Q)+1,\dots,-1,0,1,\dots,\kappa^*(Q)-1\}$, where

$$\kappa^*(Q)=\inf(i\ge0:\epsilon\alpha^*_i>Q),\qquad \kappa^*(-Q)=\sup(i\le0:\epsilon\alpha^*_i<-Q) \tag{2.40}$$

with the convention that $\inf(\emptyset)=+\infty,\sup(\emptyset)=-\infty$ and $\epsilon\alpha^*_0<0$ and $\epsilon\alpha^*_1>0$, that is just a relabeling of the points determined in [16], Section 5. The $\kappa^*(-Q)$ and $\kappa^*(Q)$ are random numbers and Lemma 5.14 gives that, with a $I\!P$–probability absorbed in (2.34), we have $|\kappa^*(-Q)|\vee\kappa^*(Q)\le K(Q)$, with $K(Q)$ given in (2.35). This implies (2.38)

2.4. The main results

Let u and $u^* \in BV_{\rm loc}$. Denote by $(W(r), r \in I\!\!R)$ the Bilateral Brownian motion (BBM) , *i.e.* the Gaussian process with independent increments that satisfies $I\!\!E(W(r)) = 0$, $I\!\!E(W^2(r)) = |r|$ for $r \in I\!\!R$ (and therefore $W(0) = 0$) and by $\mathcal{P}$ its Wiener measure on $C(I\!\!R, \mathcal{B}(C(R)))$. Let W be a real valued continuous function from $I\!\!R$ to $I\!\!R$, that is a realization of a BBM. Let $[a, b) \subset I\!\!R$ be a finite interval and denote by $\mathcal{N}_{[a,b)}(u, u^*)$ the points of jump of u or u^* in $[a, b)$:

$$\mathcal{N}_{[a,b)}(u, u^*) = \{r \in [a, b) : \|Du(r)\|_1 \neq 0 \text{ or } \|Du^*(r)\|_1 \neq 0\}. \tag{2.41}$$

Note that by right continuity if $\|Du(a)\|_1 \neq 0$ then $a \in \mathcal{N}_{[a,b)}(u, u^*)$, while if $\|Du(b)\|_1 \neq 0$ then $b \notin \mathcal{N}_{[a,b)}(u, u^*)$. Since u and u^* are $BV_{\rm loc}$ functions $\mathcal{N}_{[a,b)}(u, u^*)$ is a finite set of points. We index in increasing order the points in $\mathcal{N}_{[a,b)}(u, u^*)$ and by an abuse of notation we denote $\{i \in \mathcal{N}_{[a,b)}(u, u^*)\}$ instead of $\{i : r_i \in \mathcal{N}_{[a,b)}(u, u^*)\}$. Define for $u \in BV_{\rm loc}$, the following finite volume functional

$$\begin{aligned}&\Gamma_{[a,b)}(u|u^*, W)\\ &= \frac{1}{2\tilde{m}_\beta} \sum_{i \in \mathcal{N}_{[a,b)}(u,u^*)} \left\{ \frac{\mathcal{F}^*}{2} \left[\|Du(r_i)\|_1 - \|Du^*(r_i)\|_1 \right] - V(\beta, \theta)(\tilde{u}(r_i) - \tilde{u}^*(r_i))[W(r_{i+1}) - W(r_i)] \right\}.\end{aligned} \tag{2.42}$$

The functional in (2.42) is always well defined since it is sum of finite terms. In the following $u^* \equiv u^*(W)$ is a $BV_{\rm loc}$ function determined by the realization of the BBM. We construct it through the $h-$extrema of BBM where $h = \frac{2\mathcal{F}^*}{V(\beta,\theta)}$. In Section 5, we recall the construction done by Neveu and Pitman, [32], together with all its relevant properties. Here we only recall what is needed to state the main theorems. Denote, as in [32] , by $\{S_i \equiv S_i^{(h)}; \in Z\!\!\!Z\}$ the points of $h-$ extrema with the labeling convention that $\ldots S_{-1} < S_0 \leq 0 < S_1 < S_2 \ldots$. They proved that $\{S_i \equiv S_i^{(h)}; \in Z\!\!\!Z\}$ is a stationary renewal process, and gave the Laplace transform of the inter-arrival times. The $u^* = u^*(W)$ is the following random $BV_{\rm loc}$ function:

$$u^*(r) = \begin{cases} m_\beta, & \text{for} \quad r \in [S_i, S_{i+1}), \quad \text{if} \quad S_i \quad \text{is a point of } h\text{–minimum for } W \text{ ;} \\ Tm_\beta, & \text{for} \quad r \in [S_{i+1}, S_{i+2}). \end{cases} \tag{2.43}$$

$$u^*(r) = \begin{cases} Tm_\beta, & \text{for} \quad r \in [S_i, S_{i+1}), \quad \text{if} \quad S_i \quad \text{is a point of } h\text{–maximum for } W \text{ ;} \\ m_\beta, & \text{for} \quad r \in [S_{i+1}, S_{i+2}). \end{cases} \tag{2.44}$$

For W and $u^*(W)$ chosen as described, we denote for $u \in BV_{\rm loc}$

$$\Gamma(u|u^*, W) = \sum_{j \in Z\!\!\!Z} \Gamma_{[S_j, S_{j+1})}(u|u^*(W), W). \tag{2.45}$$

Since $|[S_j, S_{j+1})|$ is $\mathcal{P}$ a.s a finite interval, $\Gamma_{[S_j,S_{j+1})}(u|u^*(W), W)$ is well defined. Actually it can be proven, see Theorem 2.4 stated below, that the sum is positive and therefore the functional in (2.45) is well defined although it may be infinite. The $\Gamma(\cdot|u^*, W)$ provides an extension of the functional (2.42) in $I\!\!R$. One can formally write the functional (2.45) as

$$\Gamma(u|u^*, W) = \frac{1}{2\tilde{m}_\beta} \left\{ \frac{\mathcal{F}^*}{2} \int_{I\!\!R} dr \left[\|Du(r)\|_1 - \|Du^*(r)\|_1 \right] - V(\beta, \theta) \int_{I\!\!R} (\tilde{u}(r) - \tilde{u}^*(r)) dW(r) \right\}. \tag{2.46}$$

The stochastic integral in (2.46) should be defined but since we use merely (2.45) that is well defined, we leave (2.46) as a formal definition. We have the following result:

Theorem 2.4 *$\mathcal{P}$ a.s. one can construct an unique $u^* = u^*(W) \in BV_{\rm loc}$ such that for any $u \in BV_{\rm loc}$, $\Gamma(u|u^*, W) \geq 0$.*

Theorem 2.5 and Corollary 2.6 stated next relate $u^*_\gamma(\omega)$ defined in (2.39) with $u^*(W)$ the minimizer of $\Gamma(u|u^*, W)$.

Theorem 2.5 *Given $(\beta, \theta) \in \mathcal{E}$, see (2.19), choosing the parameters as in Subsection 2.5, setting $h = \frac{2\mathcal{F}^*}{V(\beta,\theta)}$, we have that*

$$\lim_{\gamma \to 0} \epsilon(\gamma)\alpha^*_i(\gamma) \stackrel{\rm Law}{=} S_i^{(h)} \equiv S_i \tag{2.47}$$

for $i \in \mathbb{Z}$. The $\{S_i, i \in \mathbb{Z}\}$ is a stationary renewal process on $\mathbb{R}$. The $S_{i+1} - S_i$, (and $S_{-i} - S_{-i-1}$) for $i > 1$ are independent, equidistributed, with Laplace transform

$$\mathbb{E}[e^{-\lambda(S_{i+1}-S_i)}] = [\cosh(h\sqrt{2\lambda})]^{-1} \text{ for } \lambda \geq 0 \tag{2.48}$$

(mean h^2) and distribution given by

$$\frac{d}{dx}\left(\mathbb{P}[S_2 - S_1 \leq x]\right) = \frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)}{h^4} \exp\left[-(2k+1)^2 \frac{\pi^2}{8}\frac{x}{h^2}\right] \quad \text{for x} > 0. \tag{2.49}$$

Moreover S_1 and $-S_0$ are equidistributed, have Laplace transform

$$\mathbb{E}[e^{-\lambda S_1}] = \frac{1}{h^2\lambda}\left(1 - \frac{1}{\cosh(h\sqrt{2\lambda})}\right) \text{ for } \lambda \geq 0 \tag{2.50}$$

and distribution given by

$$\frac{d}{dx}\left(\mathbb{P}[S_1 \leq x]\right) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)h^2} \exp\left[-(2k+1)^2 \frac{\pi^2}{8}\frac{x}{h^2}\right] \quad \text{for x} > 0. \tag{2.51}$$

The formula (2.48) was given by Neveu and Pitman in [32] and is reported here for completeness.

Corollary 2.6 *Under the same hypothesis of Theorem 2.5, for the topology induced by the Skorohod metric that makes $BV_{\rm loc}$ a complete separable space, see (5.4) we have*

$$\lim_{\gamma \downarrow 0} u^*_\gamma \stackrel{\rm Law}{=} u^*. \tag{2.52}$$

The proof of Theorem 2.5 and Corollary 2.6 are given in Section 5.

Remark 2.7 . Note that the Laplace transform (2.50) is the one of the limiting distribution of the age or the residual life of a renewal process whose Laplace transform of inter-arrival times is given is (2.48). The explicit expression given in (2.51) is the same found by H. Kesten, [26], and independently by Golosov, [22], for the limiting distribution of the point of localization of the Sinai random walk in one dimension given that this point is positive, [35]. The formula (2.49) can be easily obtained from (2.51) knowing that (2.51) is the limiting distribution of the age of the above renewal process.

Remark 2.8 . An immediate consequence of Theorems 2.3 and 2.5 is that to construct the limiting (in Law when $\gamma \downarrow 0$) typical profile of the Gibbs measure one can proceed in the following way: Starting on the right of the origin take a sample of a random variable with distribution (2.51) and put a mark there and call it S_1, make the same on the left of the origin and call the mark S_0. Then the limiting typical profile will be constant on $[S_0, S_1)$. To determine if it is equal to m_β or Tm_β, just take a sample of a symmetric Bernoulli random variable τ with value in $\{m_\beta, Tm_\beta\}$ and take the profile accordingly. To continue the construction, use the above renewal process to determine marks S_2 and S_{-1} then take for the typical limiting profile in $[S_1, S_2)$ and $[S_{-1}, S_0)$ the one with $T\tau$ defined in (2.16) with T^2 the identity, then continue.

Recall that the results of Theorem 2.3 are obtained in a probability subset Ω_1 depending on γ and u^*_γ is defined only on the interval $[-Q, +Q] \equiv [-Q(\gamma), Q(\gamma)]$ which is finite for any fixed γ, see (2.67). To our aim it is convenient to consider $u^*_\gamma \in BV([-Q, +Q])$ as an element of BV_{loc} by replacing u^*_γ by u^{*Q}_γ where for $u \in BV([-Q, +Q])$,

$$u^Q(r) = \begin{cases} u(r \wedge Q), & \text{if } r \ge 0; \\ u(r \vee (-Q)), & \text{if } r < 0. \end{cases} \tag{2.53}$$

In Theorem 2.5 we related the profile $u^*_\gamma(\omega) \in BV([-Q(\gamma), Q(\gamma)])$, (or what is the same u^{*Q}_γ in BV_{loc}) to $u^*(W) \in BV_{\mathrm{loc}}$. Next result which is a weak large deviation principle, connects the random functional $\Gamma(\cdot|u^*, W)$, (2.45), with a functional obtained when estimating the (random) Gibbs measures of the neighboorhood $\mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,+Q]}(u)$ for u belonging to a class of perturbations of $u^*_\gamma(\omega)$ that has to be specified. Let us denote for $Q \equiv Q(\gamma)$ and $f(Q)$ a positive increasing real function,

$$\mathcal{U}_Q(u^*_\gamma) = \Big\{ u \in BV_{\mathrm{loc}};\ u^Q(r) = u^{*Q}_\gamma(r), \forall\, |r| \ge Q-1,\ V^Q_{-Q}(u) \le V^Q_{-Q}(u^*_\gamma) f(Q) \Big\}. \tag{2.54}$$

The last requirement in (2.54) imposes that the number of jumps of u does not grow too fast with respect to the ones of u^*_γ. Note that u in $\mathcal{U}_{Q(\gamma)}(u^*_\gamma(\omega))$ is a random function depending on γ that is $u \equiv u(\gamma)$ and $u \equiv u(\gamma, \omega)$ if one needs to emphasize the ω dependence.

Theorem 2.9 *Given $(\beta, \theta) \in \mathcal{E}$, let $u^* \in BV_{\mathrm{loc}}$ be the $\mathcal{P}$ a.s. minimizer for $\Gamma(\cdot|u^*, W)$ given in (2.43), (2.44). Setting the parameters as in Subsection 2.5, taking*

$$f(Q) = e^{(\frac{1}{8+4a} - b)(\log Q)(\log\log Q)}, \tag{2.55}$$

with $0 < b < 1/(8+4a)$, for $u(\gamma, \omega) \in \mathcal{U}_Q(u^*_\gamma(\omega))$ such that

$$\lim_{\gamma \downarrow 0} (u(\gamma), u^*_\gamma) \stackrel{\mathrm{Law}}{=} (u, u^*) \tag{2.56}$$

we have

$$\lim_{\gamma \downarrow 0} \left[-\gamma \log \mu_{\beta,\theta,\gamma} \Big(\mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u(\gamma)) \Big) \right] \stackrel{\mathrm{Law}}{=} \Gamma(u|u^*, W). \tag{2.57}$$

Let us consider some examples: Suppose $u_1 \in \mathcal{U}_Q(u^*_\gamma(\omega))$ is such that for some $L > 0$

$$u_1(\omega)(r) = v(r) 1\!\!1_{[-L,L]}(r) + u^*_\gamma(\omega) 1\!\!1_{[-Q,Q]\setminus[-L,L]}(r), \tag{2.58}$$

where $v \in BV_{\mathrm{loc}}$ is non random function. When L is a fixed number independent on γ then (u_1, u^*_γ) converges in Law when $\gamma \downarrow 0$ to

$$(v(r) 1\!\!1_{[-L,L]}(r) + u^*(r, W) 1\!\!1_{I\!R\setminus[-L,L]}(r), u^*(r, W))$$

and the functional in the r.h.s. of (2.57) is computed on $u(r) = v(r)\mathbb{1}_{[-L,L]}(r)+u^*(r,W)\mathbb{1}_{\mathbb{R}\setminus[-L,L]}(r)$. When $L \equiv L(\gamma)$ in (2.58) goes to infinity when $\gamma \downarrow 0$ then (u_1, u^*_γ) converges in Law to (v, u^*) and the functional in the r. h. s. of (2.57) is computed on the function $u(r) \equiv v(r)$. Theorem 2.9 is a consequence of accurate estimates, see Proposition 4.2, where approximate terms and errors are explicitly computed.

2.5 Choice of the parameters We regroup here the choice of the parameters that will be used all along this work. This choice is similar to the one done in [16]. First one chooses a function g on $[1,\infty)$ such that $g(x) > 1, g(x)/x \le 1, \forall x > 1$ and $\lim_{x\uparrow\infty} x^{-1}g^{38}(x) = 0$. Any increasing function slowly varying at infinity can be modified to satisfy such constraints. A possible choice is $g(x) = 1 \vee \log x$ or any iterated of it. For δ^*, which represents the smallest coarse graining scale, we have two constraints:

$$\frac{(\delta^*)^2}{\gamma} g^{3/2}(\frac{\delta^*}{\gamma}) \le \frac{1}{\beta\kappa(\beta,\theta)e^3 2^{13}}, \tag{2.59}$$

where $\kappa(\beta,\theta)$ is the constant in (2.22) and

$$(\frac{2\gamma}{\delta^*})^{1/2}\left(\log\frac{1}{\gamma\delta^*} + \frac{\log g(\delta^*/\gamma)}{\log\log g(\delta^*/\gamma)}\right) \le \frac{1}{32}. \tag{2.60}$$

A possible choice of δ^* is

$$\delta^* = \gamma^{\frac{1}{2}+d^*} \quad \text{for some } 0 < d^* < 1/2. \tag{2.61}$$

The first constraint, (2.59), is needed to represent in a manageable form the multibody interaction that comes from the block spin transformation, see Lemma 3.5; the second one, (2.60) is needed to estimate the Lipschitz norm when applying a concentration inequality to some function of the random potential.

Taking g slowly varying at infinity, the conditions (2.61) (2.59) and (2.60) are satisfied by taking γ small enough. For (ζ,δ), the accuracy chosen to determine how close is the local magnetization to the equilibrium values, there exists a $\zeta_0 = \zeta_0(\beta,\theta)$ such that

$$\frac{1}{[\kappa(\beta,\theta)]^{1/3} g^{1/6}(\frac{\delta^*}{\gamma})} < \zeta \le \zeta_0, \tag{2.62}$$

and δ is taken as

$$\delta = \frac{1}{5(g(\frac{\delta^*}{\gamma}))^{1/2}}. \tag{2.63}$$

The fuzziness ρ is chosen as

$$\rho = \left(\frac{5}{g(\delta^*/\gamma)}\right)^{1/(2+a)}, \tag{2.64}$$

where a is an arbitrary positive number. Note that $\delta \le \rho$ and $\rho/\delta \uparrow \infty$, so in Definition 2.2 we have just a constraint of the form $\gamma \le \gamma_0(u)$. Furthermore ϵ that appears in (2.39) is chosen as

$$\epsilon = (5/g(\delta^*/\gamma))^4, \tag{2.65}$$

R_2 that appears in Definition 2.2 is chosen as

$$R_2 = c(\beta,\theta)(g(\delta^*/\gamma))^{7/2} \tag{2.66}$$

for some positive $c(\beta,\theta)$, and

$$Q = \exp[(\log g(\delta^*/\gamma))/\log\log g(\delta^*/\gamma)]. \tag{2.67}$$

Note that choosing δ^* as in (2.61), in Theorem 2.3 we have called $g(1/\gamma)$ what is really $g(\gamma^{-(1/2)+d^*})$ however since g is rather arbitrary this is the same. Note also that since g is slowly varying at infinity, as we already mentioned $\gamma R_2 \downarrow 0$ when $\gamma \downarrow 0$. At last, note that the only constraint on ζ is (2.62). In particular one can also choose ζ in Theorem 2.3 as

$$\zeta = \zeta(\gamma) \equiv 2 \frac{1}{[\kappa(\beta,\theta)]^{1/3} g^{1/6}(\frac{\delta^*}{\gamma})} \tag{2.68}$$

that goes to zero with γ.

3 The block spin representation and Basic Estimates

As explained in the introduction the first step is a coarse graining procedure. The microscopic spin system is mapped into a block spin system (macro scale), for which the length of the interaction becomes 1. In this section we state the results of this procedure, see Lemmas 3.1 and 3.2. The actual computations are straightforward, but tedious. Once this procedure has been accomplished one needs to estimate and represent in a form, convenient for further computations, the multibody interaction, which is a byproduct of the coarse graining procedure, and the main stochastic contribution to the coarse grained energy. The multibody interaction is represented as a convergent series applying a well known Statistical Mechanics technique, the Cluster expansion, see Lemma 3.5. The main stochastic term is represented with the help of Central Limit Theory, in Proposition 3.6. We then give some basic estimates which we will apply in Section 4.

With $C_{\delta^*}(V)$ as in (2.5), let $\Sigma_V^{\delta^*}$ denote the sigma–algebra of $\mathcal{S}$ generated by $m_V^{\delta^*}(\sigma) \equiv (m^{\delta^*}(x,\sigma),\ x \in \mathcal{C}_{\delta^*}(V))$, where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$, cf. (2.8). We take $I = [i^-, i^+) \subseteq I\!R$ with $i^\pm \in Z\!\!Z$. The interval I is assumed to be $\mathcal{D}_{\delta^*}$–measurable and we set $\partial^+ I \equiv \{x \in I\!R\colon i^+ \le x < i^+ + 1\}$, $\partial^- I \equiv \{x \in I\!R\colon i^- - 1 \le x < i^-\}$, and $\partial I = \partial^+ I \cup \partial^- I$. For $(m_I^{\delta^*}, m_{\partial I}^{\delta^*})$ in $\mathcal{M}_{\delta^*}(I \cup \partial I)$, cf. (2.12), we set $\widetilde{m}^{\delta^*}(x) = (m_1^{\delta^*}(x) + m_2^{\delta^*}(x))/2$,

$$E(m_I^{\delta^*}) \equiv -\frac{\delta^*}{2} \sum_{(x,y)\in\mathcal{C}_{\delta^*}(I)\times\mathcal{C}_{\delta^*}(I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y), \tag{3.1}$$

$$E(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}) \equiv -\delta^* \sum_{x\in\mathcal{C}_{\delta^*}(I)} \sum_{y\in\mathcal{C}_{\delta^*}(\partial^\pm I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y), \tag{3.2}$$

where $J_{\delta^*}(x) = \delta^* J(\delta^* x)$. Further denote

$$\begin{aligned}\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*}) =& E(m_I^{\delta^*}) + E(m_I^{\delta^*}, m_{\partial I}^{\delta^*}) - \frac{\theta\delta^*}{2} \sum_{x\in\mathcal{C}_{\delta^*}(I)} (m_1^{\delta^*}(x) - m_2^{\delta^*}(x)) \\ &- \delta^* \sum_{x\in\mathcal{C}_{\delta^*}(I)} \frac{\gamma}{\beta\delta^*} \log \left(\frac{\delta^*\gamma^{-1}/2}{\frac{1+m_1^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2} \right) \left(\frac{\delta^*\gamma^{-1}/2}{\frac{1+m_2^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2} \right),\end{aligned} \tag{3.3}$$

$$\mathcal{G}(m_I^{\delta^*}) \equiv \sum_{x\in\mathcal{C}_{\delta^*}(I)} \mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) \tag{3.4}$$

where for each $x \in \mathcal{C}_{\delta^*}(I)$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ is the cumulant generating function:

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) \equiv -\frac{1}{\beta} \log I\!E_{x,m^{\delta^*}(x)}^{\delta^*}(e^{2\beta\theta\lambda(x)\sum_{i\in D(x)}\sigma_i}), \tag{3.5}$$

of the “canonical” measure on $\{-1,+1\}^{A(x)}$, defined through

$$\mathbb{E}^{\delta^*}_{x,m^{\delta^*}(x)}(\varphi) = \frac{\sum_\sigma \varphi(\sigma) 1\!\!1_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}}{\sum_\sigma 1\!\!1_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}}, \tag{3.6}$$

the sum being over $\sigma \in \{-1,+1\}^{A(x)}$. Finally denote

$$V(m_I^{\delta^*}) \equiv V_I(m_I^{\delta^*},h) = -\frac{1}{\beta}\log \mathbb{E}_{m^{\delta^*}(I)}\Big[\prod_{\substack{x\neq y\\ x,y\in\mathcal{C}_{\delta^*}(I)\times\mathcal{C}_{\delta^*}(I)}} e^{-\beta U(\sigma_{A(x)},\sigma_{A(y)})}\Big]. \tag{3.7}$$

where

$$U(\sigma_{A(x)},\sigma_{A(y)}) = -\sum_{i\in A(x),j\in A(y)} \gamma\big[J(\gamma|i-j|) - J(\delta^*|x-y|)\big]\sigma_i\sigma_j. \tag{3.8}$$

and

$$\mathbb{E}_{m_I^{\delta^*}}[f] \equiv \frac{\sum_{\sigma_{\gamma^{-1}I}} \prod_{x_1\in\mathcal{C}_{\delta^*}(I)} 1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}} e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i} f(\sigma)}{\sum_{\sigma_{\gamma^{-1}I}} \prod_{x_1\in\mathcal{C}_{\delta^*}(I)} 1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}} e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}}. \tag{3.9}$$

Let F^{δ^*} be a $\Sigma_I^{\delta^*}$-measurable bounded function, $m_{\partial I}^{\delta^*} \in \mathcal{M}_{\delta^*}(\partial I)$ and $\mu_{\beta,\theta,\gamma}\left(F^{\delta^*}|\Sigma_{\partial I}^{\delta^*}\right)$ the conditional expectation of F^{δ^*} given the σ–algebra $\Sigma_{\partial I}^{\delta^*}$. We obtain:

Lemma 3.1

$$\mu_{\beta,\theta,\gamma}\left(F^{\delta^*} \;\middle|\; \Sigma_{\partial I}^{\delta^*}\right)(m_{\partial I}^{\delta^*}) = \frac{e^{\pm\frac{\beta}{\gamma}2\delta^*}}{Z_{\beta,\theta,\gamma,I}(m_{\partial I}^{\delta^*})}\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} F^{\delta^*}(m^{\delta^*}) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*})+\gamma\mathcal{G}(m_I^{\delta^*})+\gamma V(m_I^{\delta^*})\right\}}, \tag{3.10}$$

where equality has to be interpreted as an upper bound for $\pm = +1$ and a lower bound for $\pm = -1$, and

$$Z_{\beta,\gamma,\theta,I}(m_{\partial I}^{\delta^*}) = \sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial I}^{\delta^*})+\gamma\mathcal{G}(m_I^{\delta^*})+\gamma V(m_I^{\delta^*})\right\}}. \tag{3.11}$$

That is, up to the error terms $e^{\pm\frac{\beta}{\gamma}2\delta^*}$, we are able to describe the system in terms of the block spin variables giving a rather explicit form to the deterministic and stochastic part. The explicit derivation of Lemma 3.1 is done in Section 3 of [16]. Here we only point out that since

$$|1\!\!1_{\{\gamma|i-j|\le 1/2\}} - 1\!\!1_{\{\delta^*|x-y|\le 1/2\}}| \le 1\!\!1_{\{-\delta^*+1/2\le\delta^*|x-y|\le\delta^*+1/2\}} \tag{3.12}$$

one can estimate

$$|U(\sigma_{A(x)},\sigma_{A(y)})| \le \gamma(\frac{\delta^*}{\gamma})^2 1\!\!1_{\{1/2-\delta^*\le\delta^*|x-y|\le 1/2+\delta^*\}}. \tag{3.13}$$

Therefore, given $m_I^{\delta^*} \in \mathcal{M}_{\delta^*}(I)$, we easily obtain from (3.13)

$$\left|H(\sigma_{\gamma^{-1}I}) + \theta\sum_{i\in\gamma^{-1}I} h_i\sigma_i - \frac{1}{\gamma}E(m_I^{\delta^*})\right| = \frac{1}{\beta}\left|\log\Big[\prod_{x\in\mathcal{C}_{\delta^*}(I)}\prod_{y\in\mathcal{C}_{\delta^*}(I)} e^{-\beta U(\sigma_{A(x)},\sigma_{A(y)})}\Big]\right| \le |I|\delta^*\gamma^{-1}, \tag{3.14}$$

for $\sigma \in \{\sigma \in \gamma^{-1}I : m^{\delta^*}(x,\sigma) = m^{\delta^*}(x), \forall x \in \mathcal{C}_{\delta^*}(I)\}$. The following lemma gives an explicit integral form of the deterministic part of the block spins system. For $m \in \mathcal{T}$, $f_{\beta,\theta}(m)$ defined in (2.18), let us call

$$\begin{aligned}\tilde{\mathcal{F}}(m_I|m_{\partial I}) = &\int_I f_{\beta,\theta}(m(x))\,dx + \frac{1}{4}\int_I\int_I J(x-y)[\tilde{m}(x)-\tilde{m}(y)]^2\,dxdy \\ &+ \frac{1}{2}\int_I dx \int_{I^c} J(x-y)[\tilde{m}(x)-\tilde{m}(y)]^2\,dy.\end{aligned} \tag{3.15}$$

Lemma 3.2 *Set $m^{\delta^*}_{I\cup\partial I} \in \mathcal{M}_{\delta^*}(I \cup \partial I)$, $m(r) = m^{\delta^*}(x)$ for $r \in [x\delta^*, (x+1)\delta^*)$ and $x \in \mathcal{C}_{\delta^*}(I \cup \partial I)$, then one has*

$$|\widehat{\mathcal{F}}(m^{\delta^*}_I|m^{\delta^*}_{\partial I}) - \tilde{\mathcal{F}}(m_I|m_{\partial I}) + \frac{\delta^*}{2}\sum_{y\in\mathcal{C}_{\delta^*}(\partial I)}\left[\tilde{m}^{\delta^*}(y)\right]^2 \sum_{x\in\mathcal{C}_{\delta^*}(I)} J_{\delta^*}(x-y)| \leq |I|\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}. \tag{3.16}$$

Proof:
Using Stirling formula, see [?], we get

$$\begin{aligned}&\Big|\delta^*\sum_{x\in\mathcal{C}_{\delta^*}(I)}\frac{1}{2\beta}\left(\mathcal{I}(m^{\delta^*}_1)+\mathcal{I}(m^{\delta^*}_2)\right) - \delta^*\sum_{x\in\mathcal{C}_{\delta^*}(I)}\frac{\gamma}{\beta\delta^*}\log\begin{pmatrix}\delta^*\gamma^{-1}/2\\ \frac{1+m^{\delta^*}_1(x)}{2}\delta^*\gamma^{-1}/2\end{pmatrix}\begin{pmatrix}\delta^*\gamma^{-1}/2\\ \frac{1+m^{\delta^*}_2(x)}{2}\delta^*\gamma^{-1}/2\end{pmatrix}\Big| \\ &\qquad\leq \frac{1}{\beta}|I|\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma},\end{aligned} \tag{3.17}$$

where $\mathcal{I}(\cdot)$ is defined after (2.18). Recalling the definition of $f_{\beta,\theta}(m)$, cf. (2.18), the lemma is proven. ■

There are two random terms in (3.10): $\mathcal{G}(m^{\delta^*}_I)$, the main random contribution, and $V(m^{\delta^*}_I)$, the random expectation of the deterministic term (3.8). To treat them we will use the following classical deviation inequality for Lipschitz function of Bernoulli random variables. See [31] or [14] for a short proof.

Lemma 3.3 *Let N be a positive integer and F be a real function on $\mathcal{S}_N = \{-1,+1\}^N$ and for all $i \in \{1,\ldots,N\}$ let*

$$\|\partial_i F\|_\infty \equiv \sup_{(h,\tilde{h}):h_j=\tilde{h}_j,\forall j\neq i}\frac{\left|F(h)-F(\tilde{h})\right|}{|h_i-\tilde{h}_i|}. \tag{3.18}$$

If $I\!P$ is the symmetric Bernoulli measure and $\|\partial(F)\|^2_\infty = \sum_{i=1}^N \|\partial_i(F)\|^2_\infty$ then, for all $t > 0$

$$I\!P\left[F - I\!E(F) \geq t\right] \leq e^{-\frac{t^2}{4\|\partial(F)\|^2_\infty}} \tag{3.19}$$

and also

$$I\!P\left[F - I\!E(F) \leq -t\right] \leq e^{-\frac{t^2}{4\|\partial(F)\|^2_\infty}}. \tag{3.20}$$

When considering volumes I that are not too large, we use the following simple fact that follows from (3.4) and (3.5)

$$|\mathcal{G}(m^{\delta^*}_I)| \leq 2\theta \sup_{\sigma_I\in\{-1,+1\}^{I/\gamma}}\sum_{x\in\mathcal{C}_{\delta^*}(I)}\Big|\sum_{i\in D(x)}\sigma_i\Big| \leq 2\theta\sum_{x\in\mathcal{C}_{\delta^*}(I)}|D(x)|. \tag{3.21}$$

Lemma 3.3 implies the following rough estimate:

Lemma 3.4 (**The rough estimate**) *For all $\delta^* > \gamma > 0$ and for all positive integer p, that satisfy*

$$12(1+p)\delta^* \log\frac{1}{\gamma} \le 1 \tag{3.22}$$

there exists $\Omega_{RE} = \Omega_{RE}(\gamma, \delta^*, p) \subseteq \Omega$ with $I\!P[\Omega_{RE}] \ge 1-\gamma^2$ such that on Ω_{RE} we have:

$$\sup_{I\subseteq[-\gamma^{-p},\gamma^{-p}]} \frac{\sum_{x\in\mathcal{C}_{\delta^*}(I)} (|D(x)| - I\!E[|D(x)|])}{\sqrt{|I|}} \le \frac{\sqrt{3(1+p)}}{\gamma}\sqrt{\gamma\log\frac{1}{\gamma}} \tag{3.23}$$

and, uniformly with respect to all intervals $I \subseteq [-\gamma^{-p}, \gamma^{-p}]$,

$$\sup_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} \gamma|\mathcal{G}(m_I^{\delta^*})|| \le 2\theta\left(\frac{|I|}{2}\sqrt{\frac{\gamma}{\delta^*}} + \sqrt{3(1+p)}\sqrt{|I|\gamma\log\frac{1}{\gamma}}\right) \le 2\theta|I|\sqrt{\frac{\gamma}{\delta^*}}. \tag{3.24}$$

This lemma is a direct consequence of Lemma 3.3, since $|D(x)| = (|D(x)| - I\!E[|D(x)|]) + I\!E[|D(x)|]$, $|D(x)| = |\sum_{i\in A(x)} h_i|/2$, and $I\!E[|D(x)|] \le \frac{1}{2}\sqrt{\delta^*/\gamma}$ by Schwarz inequality. When we use the estimate (3.24), $V(m_I^{\delta^*})$ is estimated using (3.14) and one has

$$\sup_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} \gamma|V(m_I^{\delta^*})| \le \delta^*|I|. \tag{3.25}$$

However when (3.24) and (3.25) give useless results, one can use Lemma 3.3 to estimate $V(m_I^{\delta^*})$ and at some point $\|\partial_i V(m^{\delta^*})\|_\infty$ will be needed. In Theorem 8.1 in [16], with the help of the cluster expansion, we prove the following.

Lemma 3.5 *For any finite interval I, let*

$$\|\partial_i V_I\|_\infty \equiv \sup_{(h,\tilde h):h_j=\tilde h_j,\forall j\neq i} \frac{\left|V_I(m_I^{\delta^*},h) - V_I(m_I^{\delta^*},\tilde h)\right|}{|h-\tilde h|}. \tag{3.26}$$

Then, for all $\beta > 0$, for all $\delta^* > \gamma > 0$, such that

$$\frac{(\delta^*)^2}{\gamma} \le \frac{1}{6e^3\beta} \tag{3.27}$$

we have

$$\sup_{I\subseteq\mathbb{Z}}\sup_{i\in I}\|\partial_i V_I\|_\infty \le \frac{1}{\beta}\frac{S}{1-S}, \tag{3.28}$$

where $0 < S \le 6e^3\beta\frac{(\delta^*)^2}{\gamma}$.

Together with the above estimates for V_I, we need an explicit expression for $\mathcal{G}(m_I^{\delta^*})$. Since $D(x) \subseteq B^{-\lambda(x)}(x)$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$, see (3.5), depends only on one component of $m^{\delta^*}(x)$, precisely on $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$. In fact, we have

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) = -\frac{1}{\beta}\log\frac{\sum_{\sigma\in\{-1,+1\}^{B^{-\lambda(x)}(x)}} 1\!\!1_{\{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x,\sigma)=m^{\delta^*}_{\frac{3+\lambda(x)}{2}}\}} e^{2\beta\theta\lambda(x)\sum_{i\in D(x)}\sigma_i}}{\sum_{\sigma\in\{-1,+1\}^{B^{-\lambda(x)}(x)}} 1\!\!1_{\{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x,\sigma)=m^{\delta^*}_{\frac{3+\lambda(x)}{2}}\}}}, \tag{3.29}$$

since the sums over the spin configurations in $\{-1,+1\}^{B^{\lambda(x)}(x)}$ – the ones that depend on $m^{\delta^*}_{\frac{3-\lambda(x)}{2}}$– cancel out between the numerator and denominator in (3.6). The formula (3.29) is almost useless. One can think about making an expansion in $\beta\theta$ as we basically did in [14], Proposition 3.1 where $\beta\theta$ was assumed to be as small as needed. Since here we assume $(\beta,\theta)\in\mathcal{E}$, one has to find another small quantity. Looking at the term $\sum_{i\in D(x)}\sigma_i$ in (3.5) and setting

$$p(x)\equiv p(x,\omega)=|D(x)|/|B^{\lambda(x)}(x)|=2\gamma|D(x)|/\delta^*, \tag{3.30}$$

it is easy to see that for $I\subseteq I\!R$, if

$$\left(\frac{2\gamma}{\delta^*}\right)^{1/2}\log\frac{|I|}{\delta^*}\le\frac{1}{32}, \tag{3.31}$$

we have

$$I\!P\left[\sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x)>(2\gamma/\delta^*)^{\frac14}\right]\le e^{-\frac{1}{32}\left(\frac{\delta^*}{2\gamma}\right)^{\frac12}}. \tag{3.32}$$

Depending on the values of $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ has a behavior that corresponds to the classical Gaussian, Poissonian, or Binomial regimes, as explained in [14]. It turns out, see Remark 4.11 of [16], that we need accurate estimates only for those values of $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$ for which $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ is in the Gaussian regime. In this regime, applying the Central Limit Theorem, we obtain a more convenient representation of $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ which is the content of next proposition. Let $g_0(n)$ be a positive increasing real function with $\lim_{n\uparrow\infty}g_0(n)=\infty$ such that $g_0(n)/n$ is decreasing to 0 when $n\uparrow\infty$.

Proposition 3.6 *For all $(\beta,\theta)\in\mathcal{E}$, there exist $\gamma_0=\gamma_0(\beta,\theta)$ and $d_0(\beta)>0$ such that for $0<\gamma\le\gamma_0$, $\gamma/\delta^*\le d_0(\beta)$, on the set $\{\sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x)\le(2\gamma/\delta^*)^{1/4}\}$, if*

$$|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)|\le 1-\left(\frac{g_0(\delta^*\gamma^{-1}/2)}{\delta^*\gamma^{-1}/2}\vee\frac{16p(x)\beta\theta}{1-\tanh(2\beta\theta)}\right), \tag{3.33}$$

then

$$\begin{aligned}\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))=&-\frac{1}{\beta}\log\frac{\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}}{\Psi_{0,0,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}}\\&-\frac{1}{\beta}|D(x)|\Big[\log\cosh(2\beta\theta)+\log\Big(1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\Big)+\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\Big],\end{aligned} \tag{3.34}$$

where

$$\left|\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\right|\le\left(\frac{2\gamma}{\delta^*}\right)^{1/4}\frac{32\beta\theta(1+\beta\theta)}{(1-|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)|)^2(1-\tanh(2\beta\theta))} \tag{3.35}$$

and

$$\left|\log\frac{\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}}{\Psi_{0,0,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}}\right|\le\frac{18}{g_0(\delta^*\gamma^{-1}/2)}+\left(\frac{2\gamma}{\delta^*}\right)^{1/4}c(\beta\theta), \tag{3.36}$$

with

$$c(\beta\theta)=\frac{\tanh^2(2\beta\theta)(1+\tanh^2(2\beta\theta))^2}{[1-\tanh^2(2\beta\theta)]^2[1-\tanh(2\beta\theta)]^6}. \tag{3.37}$$

The proof of Proposition 3.6 is given in Proposition 3.5 of [16]. In the following we deal with quotients of quantities (partition functions) of the type (3.11) with boundary conditions that might be different between numerator and denominator. For this reason it is convenient to introduce the following notations. Let I any finite interval. We set $m^{\delta^*}_{\partial I} = (m^{\delta^*}_{\partial-I}, m^{\delta^*}_{\partial+I})$ and, see (3.11), we denote

$$Z_{\beta,\theta,\gamma,I}\left(m^{\delta^*}_{\partial-I} = m_{s_1}, m^{\delta^*}_{\partial+I} = m_{s_2}\right) \equiv Z_I^{m_{s_1},m_{s_2}} \tag{3.38}$$

where $(m_{s_1}, m_{s_2}) \in \{m_-, 0, m_+\}^2$ and for $m_{s_1} = 0$, we set in (3.11) $E(m^{\delta^*}_I, m^{\delta^*}_{\partial-I}) = 0$ while for $m_{s_2} = 0$ we set $E(m^{\delta^*}_I, m^{\delta^*}_{\partial+I}) = 0$. In a similar way, recalling (3.10), if F^{δ^*} is $\Sigma^{\delta^*}_I$–measurable we set

$$\frac{Z_I^{m_{s_1},m_{s_2}}(F^{\delta^*})}{Z_I^{m_{s_1},m_{s_2}}} \equiv \frac{\sum_{m^{\delta^*}_I \in \mathcal{M}_{\delta^*}(I)} F(m^{\delta^*}_I) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_I | m^{\delta^*}_{\partial-I} = m_{s_1} m^{\delta^*}_{\partial+I} = m_{s_2}) + \gamma \mathcal{G}(m^{\delta^*}_I) + \gamma V(m^{\delta^*}_I)\right\}}}{Z_I^{m_{s_1},m_{s_2}}}. \tag{3.39}$$

Further, let $m^{\delta^*}_\beta$ be one of the points in $\left\{-1, -1 + \frac{4\gamma}{\delta^*}, \ldots, 1 - \frac{4\gamma}{\delta^*}, 1\right\}^2$ which is closest to m_β. Let $m^{\delta^*}_{\beta,I}$ be the function which coincides with $m^{\delta^*}_\beta$ on I and vanishes outside I and $\mathcal{R}^{\delta,\zeta}(\eta)$ for $\eta = \pm 1$ the set of configurations which are close with accuracy (δ, ζ), see (2.28), to m_β when $\eta = 1$ and to Tm_β when $\eta = -1$. By definition, $|m^{\delta^*}_\beta - m_\beta| \leq 8\gamma/\delta^*$ and choosing suitable the parameters we obtain that $m^{\delta^*}_\beta$ (resp. $Tm^{\delta^*}_\beta$) is in $\mathcal{R}^{\delta,\zeta}(+1)$, (resp $\mathcal{R}^{\delta,\zeta}(-1)$). According to the results presented in Section 2, the typical configurations profiles are long runs close to one equilibrium value followed by a jump, then again long runs close to the other equilibrium value an so on. It is therefore comprehensible that the following quantities will play an important role.

$$\frac{Z_I^{0,0}(\mathbb{1}_{T\mathcal{R}^{\delta,\zeta}(\eta)})}{Z_I^{0,0}(\mathbb{1}_{\mathcal{R}^{\delta,\zeta}(\eta)})} \equiv \frac{Z_I^{0,0}(T\mathcal{R}^{\delta,\zeta}(\eta))}{Z_I^{0,0}(\mathcal{R}^{\delta,\zeta}(\eta))}. \tag{3.40}$$

Since the two minima of $f_{\beta,\theta}$, see (2.18), are m_β and Tm_β, we have $T\mathcal{R}^{\delta,\zeta}(\eta) = \mathcal{R}^{\delta,\zeta}(-\eta)$, and we write (3.40) as

$$\frac{Z_I^{0,0}(\mathcal{R}^{\delta,\zeta}(-\eta))}{Z_I^{0,0}(\mathcal{R}^{\delta,\zeta}(\eta))} \equiv e^{\beta\Delta^\eta \mathcal{G}(m^{\delta^*}_{\beta,I})} \frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))} \tag{3.41}$$

where

$$\Delta^\eta \mathcal{G}(m^{\delta^*}_{\beta,I}) \equiv \eta\left[\mathcal{G}(m^{\delta^*}_{\beta,I}) - \mathcal{G}(Tm^{\delta^*}_{\beta,I})\right] = -\eta \sum_{x \in \mathcal{C}_{\delta^*}(I)} X(x), \tag{3.42}$$

$$X(x) = \mathcal{G}_{x,m^{\delta^*}_\beta}(\lambda(x)) - \mathcal{G}_{x,Tm^{\delta^*}_\beta}(\lambda(x)), \tag{3.43}$$

and

$$\begin{aligned} &\frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))} \\ &\equiv \frac{\sum_{m^{\delta^*}_I \in \mathcal{M}^{\delta^*}(I)} \mathbb{1}_{\{\mathcal{R}^{\delta,\zeta}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_I, 0) + \gamma \Delta_0^{-\eta}\mathcal{G}(m^{\delta^*}_I) + \gamma V(Tm^{\delta^*}_I)\right\}}}{\sum_{m^{\delta^*}_I \in \mathcal{M}_{\delta^*}(I)} \mathbb{1}_{\{\mathcal{R}^{\delta,\zeta}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_I, 0) + \gamma \Delta_0^{\eta}\mathcal{G}(m^{\delta^*}_I) + \gamma V(m^{\delta^*}_I)\right\}}}, \end{aligned} \tag{3.44}$$

and

$$\Delta_0^\eta \mathcal{G}(m^{\delta^*}_I) \equiv \sum_{x \in \mathcal{C}_{\delta^*}(\tilde{I}_{12})} \Delta_0^\eta \mathcal{G}^h_{x,m^{\delta^*}(x)} \tag{3.45}$$

where recalling (3.5),

$$\Delta_0^\eta \mathcal{G}^h_{x,m^{\delta^*}(x)} = \mathcal{G}_{x,T^{\frac{1-\eta}{2}} m^{\delta^*}(x)}(\lambda(x)) - \mathcal{G}_{x,T^{\frac{1-\eta}{2}} m_\beta^{\delta^*}(x)}(\lambda(x)) \tag{3.46}$$

with T^0 equal to the identity. When flipping h_i to $-h_i$, for all i, then $\lambda(x) \to -\lambda(x)$, $B^+(x) \to B^-(x)$ while $D(x)$ does not change. Therefore,

$$\frac{Z_I^{0,0}(\mathcal{R}(-\eta))}{Z_I^{0,0}(\mathcal{R}(\eta))}(h) = \frac{Z_I^{0,0}(\mathcal{R}(\eta))}{Z_I^{0,0}(\mathcal{R}(-\eta))}(-h), \tag{3.47}$$

which implies that $\log \frac{Z_I^{0,0}(\mathcal{R}(-\eta))}{Z_I^{0,0}(\mathcal{R}(\eta))}(h)$ is a symmetric random variable, in particular has mean zero. Further the $X(x)$ in (3.43) is a symmetric random variable as it can be directly checked inspecting (3.5). Therefore $\log \frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))}(h)$ is a symmetric random variable having mean zero and it has been estimated in [16] applying Lemma 3.3. In [16] this term was denoted $\frac{Z_{-\eta,0,\delta,\zeta}(I)}{Z_{\eta,0,\delta,\zeta}(I)}$. The estimate is reported in the next Lemma.

Lemma 3.7 *Given $(\beta,\theta) \in \mathcal{E}$, there exist $\gamma_0 = \gamma_0(\beta,\theta) > 0$, $d_0 = d_0(\beta,\theta) > 0$, and $\zeta_0 = \zeta_0(\beta,\theta)$ such that for all $0 < \gamma \le \gamma_0$, for all $\delta^* > \gamma$ with $\gamma/\delta^* \le d_0$, for all $0 < \zeta < \zeta_0$ that satisfy the following condition*

$$\zeta \ge \left(5184(1+c(\beta\theta))^2 (\frac{\gamma}{\delta^*})^{1/2}\right) \vee \left(12 \frac{e^3 \beta}{c(\beta,\theta)} \frac{(\delta^*)^2}{\gamma}\right)^2 \tag{3.48}$$

where $c(\beta\theta)$ is given in (3.37) and $c(\beta,\theta)$ is another β,θ dependent constant, then for all $a > 0$,

$$I\!P\left[\max_{I\subseteq\Delta_Q}{}^{*} \max_{\tilde I_{12}\subseteq I} \left|\log \frac{Z^{0,0}_{\tilde I_{12},0}(\mathcal{R}(-\eta))}{Z^{0,0}_{\tilde I_{12},0}(\mathcal{R}(\eta))}\right| \ge \beta \frac{4a+12\zeta}{\gamma}\right] \le \frac{2Q}{\epsilon} \frac{e^{-\frac{u}{\epsilon}}}{1-e^{-\frac{u}{\epsilon}}} \tag{3.49}$$

where $\max_{I\subseteq\Delta_Q}{}^{*}$ denote the maximum over the intervals $I \subseteq \Delta_Q$ such that $|I| = \epsilon\gamma^{-1}$ and $u \equiv \frac{a^2\beta^2}{8\zeta c^2(\beta,\theta)}$.

To apply Lemma 3.3 one needs a control of the Lipschitz norm of the object to be estimated. The Lipschitz norm of $\log \frac{Z_{I,0}^{0,0}(\mathcal{R}(-\eta))}{Z_{I,0}^{0,0}(\mathcal{R}(\eta))}$ is given in Lemma 4.9 of [16]. The estimate in Lemma 3.7 holds for interval I, $|I| = \epsilon\gamma^{-1}$. To treat intervals longer than $\frac{\epsilon}{\gamma}$, see Lemma 4.19, Section 4, one needs a non trivial extension of Lemma 3.7 and a convenient choice of the parameters involved in the estimate. This was done in the proof of Lemma 6.3 of [16]. The estimate (3.49) is useful when ϵ is small ($\epsilon \to 0$). When dealing with intervals of order $\frac{1}{\gamma}$, ($\epsilon = 1$) to get an useful estimate we need to have $u \to \infty$. The only way to obtain this, see the choice of u in Lemma 3.7, is to let $\zeta(\gamma) \to 0$ as $\gamma \to 0$. But ζ is the accuracy we choose and we would like to have ζ satisfying (3.48), small but not going to zero when $\gamma \to 0$. So the main effort is to show, that eventhough the accuracy to define the vicinity of the profiles to m_β or Tm_β is kept finite, it is possible to find with overwhelming Gibbs probability and $I\!P$ a.s., blocks in which the typical magnetization profiles are indeed at distance less than ζ_5 to m_β or Tm_β, with $\zeta_5(\gamma) \to 0$ as $\gamma \to 0$. This allows to replace the ζ in the definition of u with ζ_5. This is done in Theorem 7.4 of [16] and it will be applied when proving Proposition 4.2 in Section 4.

To treat the term in (3.42) we apply Proposition 3.6 on the set $\{p(x) \le (2\gamma/\delta^*)^{1/4}\}$ and we obtain a very convenient representation for $X(x)$

$$X(x) = -\lambda(x)|D(x)| \left[\log \frac{1+m^{\delta^*}_{\beta,2}\tanh(2\beta\theta)}{1-m^{\delta^*}_{\beta,1}\tanh(2\beta\theta)} + \Xi_1(x,\beta\theta,p(x))\right] - \lambda(x)\Xi_2(x,\beta\theta,p(x)) \tag{3.50}$$

where Ξ_1 and Ξ_2 are easily obtained from (3.34). Furthermore, choosing $g_0(n) = n^{1/4}$ in Proposition 3.6, it follows that

$$|\Xi_1(x, \beta\theta, p(x))| \le 64 \frac{\beta\theta(1+\beta\theta)}{(1-m_{\beta,1})^2(1-\tanh(2\beta\theta))} (2\frac{\gamma}{\delta^*})^{1/4} \tag{3.51}$$

and

$$|\Xi_2(x, \beta\theta, p(x)| \le (2\frac{\gamma}{\delta^*})^{1/4} [36 + 2c(\beta\theta)] \tag{3.52}$$

where $c(\beta\theta)$ is given in (3.37). The $X(x)$ are in fact symmetric random variables as it follows from (3.50). We have that

$$\begin{aligned} &I\!E[X(x) 1\!\!I_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}] = 0, \\ &I\!E[X^2(x) 1\!\!I_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}] = \frac{\delta^*}{\gamma} c(\beta,\theta,\gamma/\delta^*) \end{aligned} \tag{3.53}$$

where $c(\beta, \theta, \gamma/\delta^*)$ satisfies

$$\begin{aligned} &c(\beta,\theta,\gamma/\delta^*) \le (V(\beta,\theta))^2 \left[1 + (\gamma/\delta^*)^{\frac{1}{5}}\right]^2 \\ &c(\beta,\theta,\gamma/\delta^*) \ge (V(\beta,\theta))^2 \left[1 - (\gamma/\delta^*)^{\frac{1}{5}}\right]^2 \end{aligned} \tag{3.54}$$

and $V(\beta, \theta)$ is defined in (2.36). By the results in [16], the runs of configurations close to m_β or to Tm_β are of order 1 in Brownian scale ($\frac{1}{\gamma^2}$ in micro units), so it is convenient to partition $I\!R$ into blocks of length ϵ, in the Brownian scale; i.e. each block in micro units is of length $\frac{\epsilon}{\gamma^2}$ and the basic assumption is that $\epsilon \equiv \epsilon(\gamma)$, $\lim_{\gamma\to 0} \epsilon(\gamma) = 0$, $\frac{\epsilon}{\gamma^2} > \frac{\delta^*}{\gamma}$, so that each block of length $\frac{\epsilon}{\gamma^2}$ contains at least one block $A(x)$; to avoid rounding problems we assume $\epsilon/\gamma\delta^* \in I\!N$, and that the basic initial partition $A(x)\colon x \in C_{\delta^*}(I\!R)$ is a refinement of the present one. We define for $\alpha \in Z\!\!Z$:

$$\chi^{(\epsilon)}(\alpha) \equiv \gamma \sum_{x:\delta^* x \in \tilde{A}_{\epsilon/\gamma}(\alpha)} X(x) 1\!\!I_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}, \tag{3.55}$$

where $\tilde{A}_{\epsilon/\gamma}(\alpha) = [\alpha\frac{\epsilon}{\gamma}, (\alpha+1)\frac{\epsilon}{\gamma})$ and for the sake of simplicity the γ, δ^* dependence is not explicit. To simplify further, and if no confusion arises, we shall write simply $\chi(\alpha)$. Note that $\chi(\alpha)$ is a symmetric random variable and from (3.53)

$$\begin{aligned} &I\!E[\chi(\alpha)] = 0 \\ &I\!E[\chi^2(\alpha)] = \epsilon c(\beta,\theta,\gamma/\delta^*). \end{aligned} \tag{3.56}$$

It was proved in [16], Lemma 5.4, that there exists $d_0(\beta, \theta) > 0$ such that if $\gamma/\delta^* \le d_0(\beta, \theta)$ then for all $\lambda \in I\!R$ we have

$$I\!E\left[e^{\lambda\chi(\alpha)}\right] \le e^{\frac{3\lambda^2}{4}\epsilon V^2(\beta,\theta)} \tag{3.57}$$

where $V(\beta, \theta)$ is defined in (2.36).

4 Finite volume estimates

In this section, we give upper and lower bounds of the infinite volume random Gibbs probability $\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,Q]}(u)\big)$ in term of finite volume quantities, see Proposition 4.2. This is the fundamental ingredient in the proof of Theorem 2.9. By assumption $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$, see (2.54), where for $\omega \in \Omega_1$, the probability subset in Theorem 2.3, $u^*_\gamma(\omega) \in BV([-Q,+Q])$ is the profile defined in (2.39). There is no lost of generality to assume that there exists a positive integer L, $L < Q$ such that

$$u_\gamma(r) = u^*_\gamma(r), \;\; \forall |r| \geq L. \tag{4.1}$$

To avoid the case that a jump of $u^*_\gamma(\omega)$ occurs at L or $-L$, we require that

$$\{-L\} \cup \{L\} \notin \cup_{i=\kappa^*(-Q)}^{\kappa^*(Q)} [\epsilon\alpha^*_i - 2\rho, \epsilon\alpha^*_i + 2\rho], \tag{4.2}$$

where $\kappa^*(\pm Q)$ are defined in (2.40) and ρ is chosen as in (2.64). To see that requirement (4.2) is harmless, let

$$\Omega_3 \equiv \Omega_3(Q) = \bigcup_{L\in[1,Q]\cap \mathbb{Z}} \left\{\omega : \{-L\} \cup \{L\} \in \cup_{i=k(-Q)}^{k(Q)} [\epsilon\alpha^*_i - 2\rho, \epsilon\alpha^*_i + 2\rho]\right\}. \tag{4.3}$$

We have the following result.

Lemma 4.1 *There exist $\gamma_0(\beta,\theta) > 0$ and $a > 0$ such that for $\gamma \leq \gamma_0 = \gamma_0(\beta,\theta)$ we have*

$$I\!P[\Omega_3] \leq \frac{Q}{(g(\frac{\delta^*}{\gamma}))^{\frac{1\wedge a}{8(2+a)}}} \leq \frac{1}{(g(\frac{\delta^*}{\gamma}))^{\frac{1\wedge a}{10(2+a)}}}\,. \tag{4.4}$$

Proof: Note that

$$\Omega_3 \subset \bigcup_{L\in[1,Q]\cap \mathbb{Z}} \{\exists i \in \{\kappa^*(-Q),\dots,\kappa^*(Q)\}, \quad \epsilon\alpha^*_i \in [L-2\rho, L+2\rho] \cup [-L-2\rho, -L+2\rho]\}\,. \tag{4.5}$$

To estimate the probability of the event (4.5), we use Lemma 5.14 where it is proven that uniformly with respect to Q and with $I\!P$–probability larger than $1 - (5/g(\delta^*/\gamma))^{\frac{a}{8(2+a)}}$, $\kappa^*(Q)$ and $\kappa^*(-Q)$ are bounded by $K(Q)$ given in (2.35). The other ingredient is the estimate of the probability that $\epsilon\alpha^*_0$ or $\epsilon\alpha^*_1 \in [-2\rho, +2\rho]$. This is done in Theorem 5.1 of [16] (see formula 5.29, 5.30 and 6.66 of [16]). Then for some $c(\beta,\theta)$, $a > 0$, when $\gamma \leq \gamma_0(\beta,\theta)$ we have the following:

$$\begin{aligned} &I\!P\left[\exists i \in \{\kappa(^*-Q),\dots,\kappa^*(Q)\} : \epsilon\alpha^*_i \in [L-2\rho, L+2\rho]\right] \\ &\leq 2c(\beta,\theta)K(Q)[g(\delta^*/\gamma)]^{-1/(4(2+a))} + \left(\frac{5}{g(\frac{\delta^*}{\gamma})}\right)^{\frac{a}{8(2+a)}} \leq \frac{1}{(g(\frac{\delta^*}{\gamma}))^{\frac{1\wedge a}{8(2+a)}}}. \end{aligned} \tag{4.6}$$

By subadditivity one gets (4.4), recalling that $Q = \exp\left[\log(g(\frac{\delta^*}{\gamma}))/\log\log(g(\frac{\delta^*}{\gamma}))\right]$. $\blacksquare$

¿From now on, we will always consider $\omega \in \Omega_1 \setminus \Omega_3$ and since the union is over $L \in [1,Q] \cap \mathbb{Z}$ in (4.3), this probability set is the same for all $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$. For $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$ and L so that (4.1) holds, denote

$$r_1 = \inf(r : r > -Q, \, \|Du(r) - Du^*_\gamma(r)\|_1 > 0); \qquad r_{\text{last}} = \sup(r : r < Q, \, \|Du(r) - Du^*_\gamma(r)\|_1 > 0) \tag{4.7}$$

where Du is defined before (2.31). The r_1 is the first point starting from $-Q$ where u differs from $u^*_\gamma(\omega)$ and r_{last} is the last point smaller than Q where u differs from $u^*_\gamma(\omega)$. We denote by

$$r_i \qquad i = 1,..N_1, \qquad r_{N_1} \equiv r_{\text{last}} \tag{4.8}$$

the points of jumps of u or $u^*_\gamma(\omega)$, between $r_1 > -Q$ and $r_{\text{last}} < Q$, in increasing order. Note that r_i could be a point of jump for both u and $u^*_\gamma(\omega)$ and

$$N_1 \le N_{[-L,+L]}(u) + N_{[-L,+L]}(u^*_\gamma(\omega)). \tag{4.9}$$

We have the following result.

Proposition 4.2 *Take the parameters as in Subsection 2.5. Let Ω_1 be the probability subspace of Theorem 2.3 and Ω_3 defined in (4.3), Ω_4 defined in Corollary 4.14 and Ω_5 defined in (4.96). On $\Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5)$, with $I\!P[\Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5))] \ge 1 - 3(g(\frac{\delta^*}{\gamma}))^{-\frac{1\wedge a}{10(2+a)}}$, for some $a > 0$, for all $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$ such that*

$$N_{[-Q,+Q]}(u) \le N_{[-Q,+Q]}(u^*_\gamma) e^{(\frac{1}{8+4a} - b)(\log Q)(\log\log Q)} \tag{4.10}$$

for $0 < b < 1/(8+4a)$, there exists a $\gamma_0 = \gamma_0(\beta,\theta,u)$ such that for all $0 < \gamma \le \gamma_0(\beta,\theta,u)$, we have that

$$\begin{aligned} &\frac{\gamma}{\beta} \log \left[\mu_{\beta,\theta,\gamma}\Big(\mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u) \Big) \right] = \\ &- \mathcal{F}^* \sum_{i=1}^{N_1} \left[\frac{\|Du(r_i)\|_1 - \|Du^*_\gamma(r_i)\|_1}{4\tilde m_\beta} \right] + \sum_{i=1}^{N_1} \frac{\tilde u(r_i) - \tilde u^*_\gamma(r_i)}{2\tilde m_\beta} \left[\sum_{\alpha:\, \epsilon\alpha \in [r_i, r_{i+1})} \chi(\alpha) \right] \pm g(\delta^*/\gamma)^{-b} \end{aligned} \tag{4.11}$$

we have an upper bound for $\pm = +1$ and a lower bound for $\pm = -1$.

Since the proof of Proposition 4.2 is rather long, we divide it in several intermediate steps. It is convenient to state the following definitions.

Definition 4.3 Partition associated to a couple (u, v) of $BV([a, b])$. *Let u and v be in $BV([a, b])$. We associate to (u, v) the partition of $[a, b]$ obtained by taking $C(u, v) = C(u) \cup C(v)$ and $B(u, v) = [a, b] \setminus C(u, v)$. The $C(u)$ and $C(v)$ are the elements of the partitions in Definition 2.2. We set $C(u, v) = \cup_{i=1}^{\bar N_{[a,b]}} C_i(u, v)$, where $\bar N_{[a,b]} \equiv \bar N(u, v, [a, b])$ is the number of disjoint intervals in $C(u, v)$, $\max\{N_{[a,b]}(u), N_{[a,b]}(v)\} \le \bar N_{[a,b]} \le N_{[a,b]}(u) + N_{[a,b]}(v)$.*

By definition, for $i \ne j$, $C_i(u) \cap C_j(u) = \emptyset$ and $C_i(v) \cap C_j(v) = \emptyset$, however when u and v have jumps at distance less than ρ, $C_i(u) \cap C_j(v) \ne \emptyset$ for some $i \ne j$ and in this case one element of $C(u, v)$ is $C_i(u) \cup C_j(v)$.

Remark 4.4 . The condition that ρ and δ are small enough in such a way that the distance between two successive jumps of u or v is larger than $8\rho + 8\delta$, see Definition 2.2, implies that the distance between any two distinct $C_i(u, v)$ is at least $2\rho + 2\delta$. This means that in a given $C_i(u, v)$ there is at most two jumps, one of u and the other of v.

The partition in Definition 4.3 induces a partition on the rescaled (macro) interval $\frac{1}{\gamma}[a, b] = C_\gamma(u, v) \cup B_\gamma(u, v)$ where $C_\gamma(u, v) = \cup_{i=1}^{\bar N_{[a,b]}} C_{i,\gamma}(u, v)$ and $C_{i,\gamma}(u, v) = \gamma^{-1} C_i(u, v)$.

We will use Definition 4.3 for the couple $(u, u^*_\gamma(\omega))$ for $u \in \mathcal{U}_Q(u^*_\gamma(\omega))$, $[a, b] = [r_1, r_{\text{last}}]$, see (4.7). For simplicity we denote

$$\bar N(u, u^*_\gamma(\omega), [r_1, r_{\text{last}}]) \equiv \bar N.$$

Of course, $N_1 \geq \bar{N}$, see (4.9). We write in macroscale

$$C_\gamma(u, u^*_\gamma) = \cup_{i=1}^{\bar{N}} [a_i, b_i), \quad [a_i, b_i) \cap [a_j, b_j) = \emptyset \qquad 1 \leq i \neq j \leq \bar{N}. \tag{4.12}$$

Remark 4.5 . In Proposition 4.2, $\gamma_0 = \gamma_0(\beta, \theta, u)$ depends on u since $8\rho(\gamma) + 8\delta(\gamma)$ has to be smaller than the distance between two successive jumps of u.

Since the estimates to prove Proposition 4.2 are done in intervals written in macroscale we make the following convention:

$$\begin{aligned} & m(x) = u(\gamma x), \quad m^*(x) = u^*_\gamma(\gamma x) \quad \text{for} \quad x \in \frac{1}{\gamma}[-Q, Q] \\ & q_1 = -\frac{Q}{\gamma}, \quad q_2 = \frac{Q}{\gamma}; \quad v_1 = -\frac{L}{\gamma}, \quad v_2 = \frac{L}{\gamma}; \quad x_i = \frac{r_i}{\gamma}, \quad i = 1, \dots, N_1 \\ & \mathcal{P}^\rho_{[q_1,q_2]}(m) \equiv \mathcal{P}^\rho_{\delta,\gamma,\zeta,[-Q,Q]}(u_\gamma), \end{aligned} \tag{4.13}$$

where we recall that r_i are the points where u or u^*_γ jumps, see (4.8). Furthermore, let us define

$$\eta(\ell, v) = \begin{cases} 0 & \text{if} \quad \ell \in C_\gamma(v); \\ 1 & \text{when } m(x) \text{ equal to } m_\beta \text{ for } x \in B_\gamma(v); \\ -1 & \text{when } m(x) \text{ equal to } Tm_\beta \text{ for } x \in B_\gamma(v). \end{cases}$$

Note that $\eta(\ell, v)$ is associated to the function v not to a block-spin configuration.

Definition 4.6 *For δ and ζ positive, for two integers $p_1 < p_2$ define*

$$\mathcal{O}_0^{\delta,\zeta}([p_1, p_2]) \equiv \left\{\eta^{\delta,\zeta}(\ell) = 0, \, \forall \ell \in [p_1, p_2]\right\} \tag{4.14}$$

and for $\bar{\eta} \in \{-1, +1\}$,

$$\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2]) \equiv \left\{\eta^{\delta,\zeta}(\ell) = \bar{\eta}, \, \forall \ell \in [p_1, p_2]\right\}. \tag{4.15}$$

The first set $\mathcal{O}_0^{\delta,\zeta}([p_1, p_2])$ contains configurations for which the block spin variable, see (2.26), is ζ far from the equilibrium values. The second set $\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2])$ contains configurations for which the block spin variable is ζ close to the equilibrium value m_β when $\bar{\eta} = 1$ or Tm_β when $\bar{\eta} = -1$.

Using a simple modification of the rather involved proof of Theorem 7.4 in [16] one gets the following.

Proposition 4.7 *Take the parameters as in Subsection 2.5. Let $\Omega_1 \setminus \Omega_3$ be the probability subset with Ω_1 in Theorem 2.3 and Ω_3 defined in (4.3). There exist $\gamma_0 = \gamma_0(\beta, \theta)$ and ζ_0 such that for all $\omega \in \Omega_1$, for all $\bar{\eta} \in \{-1, +1\}$, for all $\ell_0 \in I\!N$, for all δ, ζ, ζ_5 with $1 > \delta > \delta^* > 0$, and any $\zeta_0 > \zeta > \zeta_5 \geq 8\gamma/\delta^*$, for all $[\bar{p}_1, \bar{p}_2] \subset [p_1, p_2] \subset [q_1, q_2]$ with $\bar{p}_1 - p_1 \geq \ell_0$, $p_2 - \bar{p}_2 \geq \ell_0$, we have*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2]) \cap \mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\right) \leq e^{-\frac{\beta}{\gamma}\left\{(\bar{p}_2 - \bar{p}_1)\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3 - 48(1+\theta)\sqrt{\frac{\gamma}{\delta^*}}\right) - 2\zeta e^{-\alpha(\beta,\theta,\zeta_0)2\ell_0} - 4\ell_0\sqrt{\frac{\gamma}{\delta^*}}\right\}}. \tag{4.16}$$

Here $\alpha(\beta, \theta, \zeta_0)$ is a strictly positive constant for all $(\beta, \theta) \in \mathcal{E}$, $\kappa(\beta, \theta)$ is the same as in (2.22). Moreover

$$\sup_{[p_1,p_2] \subseteq [-\gamma^{-p}, \gamma^{-p}]} \frac{Z^{0,0}_{[p_1,p_2]}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2]) \cap \mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\right)}{Z^{0,0}_{[p_1,p_2]}\left(\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2])\right)} \tag{4.17}$$

satisfies the same estimates as (4.16).

Remark 4.8 . The terms with $\sqrt{\gamma/\delta^*}$ in the right hand side of (4.16) comes from the rough estimates see Lemma 3.4. The fact that $\alpha(\beta,\theta,\zeta_0) > 0$ is a consequence of (2.21). The term $\kappa(\beta,\theta)\delta\zeta_5^3$ comes from estimating the contribution of (2.22) for spin configuration in $\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1,\bar{p}_2])$.

We have the following.

Lemma 4.9 (reduction to finite volume) *Under the same hypothesis of Proposition 4.2 and on the probability space $\Omega_1 \setminus \Omega_3$, for ζ_5 that satisfies*

$$\delta\zeta_5^3 \geq 384(1+\zeta\frac{\gamma}{\delta^*}+\theta)\frac{1}{\kappa(\beta,\theta)\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\log\frac{\delta^*}{\gamma}, \tag{4.18}$$

we have

$$\begin{aligned}\mu^{\omega}_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\Big) \geq e^{-\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\Big(1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}\Big)\times \\ \frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\mu^{\omega}_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\Big) \leq 2e^{-\frac{\beta}{\gamma}\left\{L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3\right\}}+e^{\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\times \\ \sum_{\substack{v_1-L_1-1\leq n_0'\leq v_1\\ v_2\leq n'_{\bar{N}+1}\leq v_2+L_1+1}}\frac{Z^{0,0}_{[n_0',n'_{\bar{N}+1}]}\Big(\mathcal{P}^{\rho}_{[n_0',n'_{\bar{N}+1}]}(m),\eta^{\delta,\zeta_5}(n_0')=\eta(n_0',m^*),\eta^{\delta,\zeta_5}(n'_{\bar{N}+1})=\eta(n'_{\bar{N}+1},m^*)\Big)}{Z^{0,0}_{[n_0',n'_{\bar{N}+1}]}\Big(\mathcal{P}^{\rho}_{[n_0',n'_{\bar{N}+1}]}(m^*),\eta^{\delta,\zeta_5}(n_0')=\eta(n_0',m^*),\eta^{\delta,\zeta_5}(n'_{\bar{N}+1})=\eta(n'_{\bar{N}+1},m^*)\Big)},\end{aligned} \tag{4.20}$$

where L_1 satisfies $L_1+\ell_0 \leq \rho/\gamma$ and

$$\ell_0 = \frac{\log(\delta^*/\gamma)}{\alpha(\beta,\theta,\zeta_0)}. \tag{4.21}$$

Proof: Recalling (4.13) and (4.1), for $\omega \in \Omega_1 \setminus \Omega_3$ see (4.3), one has

$$\eta(\ell,m)=\eta(\ell,m^*)\neq 0 \quad \text{for} \quad \ell\in[v_1-\frac{\rho}{\gamma},v_1+\frac{\rho}{\gamma}]\cup[v_2-\frac{\rho}{\gamma},v_2+\frac{\rho}{\gamma}]. \tag{4.22}$$

Therefore the spin configurations in $\mathcal{P}^{\rho}_{[q_1,q_2]}(m)$ satisfy

$$\begin{aligned}&\eta^{\delta,\zeta}(\ell)(\sigma)=\eta(\ell,m)=\eta(\ell,m^*)=\eta(v_1,m^*)\neq 0;\ \forall\ell\in[v_1-\frac{\rho}{\gamma},v_1+\frac{\rho}{\gamma}]\\ &\text{and}\quad \eta^{\delta,\zeta}(\ell)(\sigma)=\eta(\ell,m)=\eta(\ell,m^*)=\eta(v_2,m^*)\neq 0;\ \forall\ell\in[v_2-\frac{\rho}{\gamma},v_2+\frac{\rho}{\gamma}].\end{aligned} \tag{4.23}$$

We start proving the lower bound (4.19). Within the proof we present a fundamental procedure, the *cutting* which allows us to estimate the infinite volume Gibbs measure with finite volume quantities. This procedure will be constantly used in the following. We explain here in details, referring to it when needed.

The lower bound

We just impose extra constraints at $v_1 - 1$ and $v_2 + 1$, that is

$$\mu_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\Big) \geq \mu_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m), \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*), \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\Big). \quad (4.24)$$

As mentioned in Section 2.1 there is an unique infinite volume Gibbs measure that can be obtained as the weak limit of finite volume Gibbs measure with 0 boundary conditions. So to estimate the infinite volume Gibbs measure in (4.24) we start considering the Gibbs measure in a volume $[-a, a]$ with $a > 0$ big enough so that $[q_1, q_2] \subset [-a, a]$. We write

$$\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m), \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*), \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[-a,a]}}. \quad (4.25)$$

The goal is to estimate (4.25) uniformly with respect to a. This will be achieved by *cutting* at $v_1 - 1$ and $v_2 + 1$, which we explain now. Divide the interval $[-a, a]$ in three pieces $[-a, v_1 - 2]$,$[v_1 - 1, v_2 + 1]$, and $[v_2 + 2, a]$. Then, associate the interaction between the first and the second interval to the first interval, and the one between the second and the third to the third interval. Use (3.14) with $I = [v_1 - 2, v_1]$ to make the block spin transformation there, this will give an error term $\beta 2\delta^*/\gamma$. Use $\eta^{\delta,\zeta_5}(v_1 - 1) \neq 0$ to get that for all configurations σ

$$\Big|E(m^{\delta^*}_{[v_1-2,v_1-1)}(\sigma), m^{\delta^*}_{[v_1-1,v_1)}(\sigma')) - E(m^{\delta^*}_{[v_1-2,v_1-1)}(\sigma), T^{\frac{1-\eta(v_1-1,m)}{2}} m^{\delta^*}_{\beta,[v_1-1,v_1)})\Big| \leq \zeta_5 \quad (4.26)$$

for σ' such that $\eta^{\delta,\zeta_5}(v_1 - 1)(\sigma'_{[v_1-1,v_1)}) = \eta^{\delta,\zeta_5}(v_1 - 1)$ where $m^{\delta^*}_{\beta}$ is defined after (3.39). Therefore one sees that up to an error $e^{\pm\frac{\beta}{\gamma}(2\delta^*+\zeta_5)}$ we can replace in (4.25) the σ, σ' interaction between $[-a, v_1-2]$ and $[v_1-1, a]$ by an interaction between σ and a constant profile $T^{\frac{1-\eta(v_1-1,m)}{2}} m^{\delta^*}_{\beta,[v_1-1,v_1)}$. Making similar computations in the intervals $[v_2, v_2 + 1)$, $[v_2 + 2, a)$ and recalling (3.39), one gets

$$\begin{aligned}
&\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m),\, \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*), \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[-a,a]}} \\
&= e^{\pm\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)} \frac{1}{Z^{0,0}_{[-a,a]}} Z^{0,m^*}_{[-a,v_1-2]}\Big(\mathcal{P}^{\rho}_{[q_1,v_1-2]}(m^*)\Big)\, Z^{m^*,0}_{[v_2+2,a]}\Big(\mathcal{P}^{\rho}_{[v_2+2,q_2]}(m^*)\Big) \times \\
&\quad \times Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\, \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*),\, \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\Big).
\end{aligned} \quad (4.27)$$

In the first term on the right hand side of (4.27), using (4.1), one has $\mathcal{P}^{\rho}_{[q_1,v_1-2]}(m^*) = \mathcal{P}^{\rho}_{[q_1,v_1-2]}(m)$ and $\mathcal{P}^{\rho}_{[v_2+2,q_2]}(m^*) = \mathcal{P}^{\rho}_{[v_2+2,q_2]}(m)$. Furthermore the boundary condition $Z^{0,m^*}_{[-a,v_1-2]}(\cdot)$ is written in term of m^*, but since on $v_1 - 1$ we have $\eta(v_1 - 1, m) = \eta(v_1 - 1, m^*)$ we could have also written $Z^{0,m}_{[-a,v_1-2]}(\cdot)$. Similar considerations hold for the partition function in $[v_2 + 2, a]$. The above procedure which allows to factorize the partition function up to some minor error, see (4.27), will be denoted *cutting at* $v_1 - 1$ *and* $v_2 + 1$.

Remark 4.10 . To perform a cutting at some point ℓ and to get an error term $e^{\pm\frac{\beta}{\gamma}(2\delta^*+\zeta_5)}$, one needs to have $\eta^{\delta,\zeta_5}(\ell) \neq 0$ at this point. Trying to cut at a point ℓ where $\eta^{\delta,\zeta}(\ell) = 0$ gives an error term $e^{\frac{\beta}{\gamma}(2\delta^*+1)}$ that will definitively ruin all future estimates. Trying to cut at a point ℓ where $\eta^{\delta,\zeta}(\ell) \neq 0$ gives an error term $e^{\frac{\beta}{\gamma}(2\delta^*+\zeta)}$. Since we are not imposing that ζ goes to zero, this will also ruin all the future estimates.

Multiplying and dividing (4.27) by

$$Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big) \tag{4.28}$$

and then regrouping, one gets

$$\begin{aligned}
&\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[-a,a]}}\geq\\
&\geq e^{-\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}\frac{1}{Z^{0,0}_{[-a,a]}}Z^{0,m^*}_{[-a,v_1-2]}\Big(\mathcal{P}^{\rho}_{[q_1,v_1-2]}(m^*)\Big)\,Z^{m^*,0}_{[v_2+2,a]}\Big(\mathcal{P}^{\rho}_{[v_2+2,q_2]}(m^*)\Big)\times\\
&\quad\times Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)\times\\
&\quad\times\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}.
\end{aligned} \tag{4.29}$$

The last term in (4.29) is the main contribution to the lower bound stated in (4.19). So to complete the proof we need to estimate from below the remaining terms in (4.29). To achieve this we do the conceptual opposite procedure of cutting. Namely we *glue* the first three partitions function in the right hand side of (4.29) at v_1-1 and v_2+1, applying again (4.26). That is

$$\begin{aligned}
&\frac{1}{Z^{0,0}_{[-a,a]}}Z^{0,m^*}_{[-a,v_1-2]}\Big(\mathcal{P}^{\rho}_{[q_1,v_1-2]}(m^*)\Big)\,Z^{m^*,0}_{[v_2+2,a]}\Big(\mathcal{P}^{\rho}_{[v_2+2,q_2]}(m^*)\Big)\times\\
&\quad\times Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)\\
&\geq e^{-\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}\times\\
&\quad\frac{1}{Z^{0,0}_{[-a,a]}}Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big).
\end{aligned} \tag{4.30}$$

By definition (2.33) and assumption, see (4.22), when $\sigma\in\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*)$ $\eta^{\delta,\zeta}(v_1-1,\sigma)\neq 0$ and $\eta^{\delta,\zeta}(v_2+1,\sigma)\neq 0$, then

$$\begin{aligned}
\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*)=&\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=0,\eta^{\delta,\zeta_5}(v_2+1)=0\Big)\cup\\
&\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big).
\end{aligned} \tag{4.31}$$

Therefore taking the limit when $a\uparrow\infty$ in the right hand side of (4.30), using Theorem 2.3 and Proposition 4.7 with $\ell_0=\frac{\log(\delta^*/\gamma)}{\alpha(\beta,\theta,\zeta_0)}$, $\bar p_2-\bar p_1=1$, $p_2-p_1>2\ell_0$ one gets

$$\begin{aligned}
&\lim_{a\uparrow\infty}\frac{1}{Z^{0,0}_{[-a,a]}}Z^{0,0}_{[-a,a]}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\big)\\
&\geq 1-K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3-48(1+\theta)\sqrt{\frac{\gamma}{\delta^*}}\frac{\log\frac{\delta^*}{\gamma}}{\alpha(\beta,\theta,\zeta_0)}\right)},
\end{aligned} \tag{4.32}$$

where $K(Q)$ is given in (2.35). Collecting (4.27), (4.29), and (4.30), using (4.18) one gets

$$\begin{aligned}
&\mu^{\omega}_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\Big)\geq e^{-\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\Big(1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}\Big)\times\\
&\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\ \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m^*),\ \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m^*)\Big)}
\end{aligned} \tag{4.33}$$

which is (4.19).

The upper bound In the proof of the lower bound we could *cut* making an error proportional to ζ_5 by simply restricting to those configurations having magnetization close to the equilibrium values with accuracy (δ, ζ_5) in the chosen $[\ell, \ell+1)$ block . In the upper bound obviously this procedure cannot be applied. We need to find a block where the spin configurations have magnetization close to the equilibrium values with accuracy (δ, ζ_5). This makes notations more cumbersome. To facilitate the reading, we use indexes with a $'$ to denote the points ℓ where $\eta^{\delta,\zeta_5}(\ell) \neq 0$. We search these points within the intervals $[v_1 - L_1 - 1, v_1 - 1]$ and $[v_2 - 1, v_2 + L_1 + 1]$ where L_1 is an integer which will be suitable chosen. ¿From (4.23) we have that $\eta^{\delta,\zeta}(\ell)(\sigma) = \eta(\ell, m^*) = \eta(v_1, m^*) \neq 0$ for $\ell \in [v_1 - L_1 - 1, v_1 - 1]$ and $\eta^{\delta,\zeta}(\ell)(\sigma) = \eta(\ell, m^*) = \eta(v_2, m^*) \neq 0$ $\ell \in [v_2 - 1, v_2 + L_1 + 1]$, provided $L_1 < \frac{\rho}{\gamma}$. Then we apply Proposition 4.7 in both the intervals, taking ℓ_0 as in (4.21) and setting

$$\begin{aligned} &p_1 = v_1 - L_1 - \ell_0, \ p_2 = v_1 + \ell_0, \\ &\bar{p}_1 = v_1 - L_1 - 1, \qquad \bar{p}_2 = v_1 \end{aligned} \tag{4.34}$$

for some L_1 such that $0 < L_1 + \ell_0 \leq \rho/\gamma$ to be chosen later. We have $\bar{p}_2 - \bar{p}_1 = L_1 + 1$ and

$$\begin{aligned} \mu_{\beta,\theta,\gamma}\Big(\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_1, p_2]) \cap \mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\Big) &\leq e^{-\frac{\beta}{\gamma}\left\{(L_1+1)\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3 - 48(1+\zeta\frac{\gamma}{\delta^*}+\theta)\sqrt{\frac{\gamma}{\delta^*}}\right) - 4\frac{\log(\delta^*/\gamma)}{\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\right\}} \\ &\leq e^{-\frac{\beta}{\gamma}\left\{L_1 \frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3\right\}}. \end{aligned} \tag{4.35}$$

At the last step, we have used (4.18). We do similarly in the interval $[v_2, v_2 + L_1 + 1]$. We apply Proposition 4.7 setting

$$\begin{aligned} &p_3 = v_2 - \ell_0, \ p_4 = v_2 + L_1 + \ell_0 \\ &\bar{p}_3 = v_2, \ \bar{p}_4 = v_2 + L_1 + 1, \end{aligned} \tag{4.36}$$

then one gets that $\mu_{\beta,\theta,\gamma}\big(\mathcal{R}^{\delta,\zeta}(\bar{\eta}, [p_3, p_4]) \cap \mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_3, \bar{p}_4])\big)$ satisfies the same estimate as in (4.35). Therefore one has the basic estimate

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big) \leq \mu_{\beta,\theta,\gamma}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m) \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\big)^c \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_3, \bar{p}_4])\big)^c\Big) + 2e^{-\frac{\beta}{\gamma}\left\{L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3\right\}}. \tag{4.37}$$

In the set $\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m) \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\big)^c \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_3, \bar{p}_4])\big)^c\big)$ there exists at least one block variable indexed by n'_0 with $\bar{p}_1 \leq n'_0 \leq \bar{p}_2$ such that $\eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m^*)$ and one block variable indexed by $n'_{\bar{N}+1}$, $\bar{p}_3 \leq n'_{\bar{N}+1} \leq \bar{p}_4$ where $\eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m^*)$. These are the blocks where we will cut. Consider the Gibbs measure in a volume $[-a, a]$ with $a > 0$ large enough so that $[q_1, q_2] \subset [-a, a]$. We have the simple estimate

$$\begin{aligned} &\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]} \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_1, \bar{p}_2])\big)^c \cap \big(\mathcal{O}_0^{\delta,\zeta_5}([\bar{p}_3, \bar{p}_4])\big)^c\Big)}{Z^{0,0}_{[-a,a]}} \leq \\ &\sum_{\substack{\bar{p}_1 \leq n'_0 \leq \bar{p}_2 \\ \bar{p}_3 \leq n'_{\bar{N}+1} \leq \bar{p}_4}} \frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m), \ \eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m), \ \eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m)\Big)}{Z^{0,0}_{[-a,a]}}. \end{aligned} \tag{4.38}$$

Consider now a generic term in the sum in the right hand side of (4.38). Recalling (4.27) and cutting at n'_0

and $n'_{\bar N+1}$, in the numerator we get

$$\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m),\, \eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m)\Big)}{Z^{0,0}_{[-a,+a]}} \le e^{\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}\frac{1}{Z^{0,0}_{[-a,a]}}$$
$$\times\, Z^{0,m^*}_{[-a,n'_0-1]}(\mathcal{P}^{\rho}_{[q_1,n'_0-1]}(m^*))\times$$
$$\times\, Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)Z^{m^*,0}_{[n'_{\bar N+1}+1,a]}(\mathcal{P}^{\rho}_{[n'_{\bar N+1}+1,q_2]}), \tag{4.39}$$

see (3.39) to recall notations. Multiplying and dividing by

$$Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m^*)\,,\eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)$$

one gets, after regrouping the terms

$$\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m)\Big)}{Z^{0,0}_{[-a,a]}} \le e^{\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}\times$$
$$\frac{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m^*),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}\times \quad .$$
$$\frac{Z^{0,m^*}_{[-a,n'_0-1]}(\mathcal{P}^{\rho}_{[q_1,n'_0-1]}(m^*))}{Z^{0,0}_{[-a,a]}}\times$$
$$Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m^*),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)Z^{m^*,0}_{[n'_{\bar N+1}+1,a]}(\mathcal{P}^{\rho}_{[n'_{\bar N+1}+1,q_2]}) \tag{4.40}$$

Now, glueing at n'_0 and $n'_{\bar N+1}$, as in (4.30), uniformly with respect to a, we have

$$\frac{Z^{0,m^*}_{[-a,n'_0-1]}(\mathcal{P}^{\rho}_{[q_1,n'_0-1]}(m^*))}{Z^{0,0}_{[-a,a]}}\times$$
$$Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m^*),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)\times \tag{4.41}$$
$$Z^{m^*,0}_{[n'_{\bar N+1}+1,a]}(\mathcal{P}^{\rho}_{[n'_{\bar N+1}+1,q_2]}) \le e^{\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}\frac{Z^{0,0}_{[-a,+a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m^*),\Big)}{Z^{0,0}_{[-a,a]}} \le e^{\frac{\beta}{\gamma}(2\zeta_5+4\delta^*)}.$$

From (4.40) and (4.41) we get

$$\frac{Z^{0,0}_{[-a,a]}\Big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m),\, \eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m)\Big)}{Z^{0,0}_{[-a,a]}} \le e^{\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\times$$
$$\times\, \frac{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\, \eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}{Z^{0,0}_{[n'_0,n'_{\bar N+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar N+1}]}(m^*),\, \eta^{\delta,\zeta_5}(n'_0)=\eta(n'_0,m^*),\, \eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(n'_{\bar N+1},m^*)\Big)}. \tag{4.42}$$

Collecting (4.38) and (4.42), one get (4.20). This ends the proof of the lemma. ∎

The configurations in $\mathcal{P}^{\rho}_{[v_1,v_2]}(m)$ and $\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*)$ are long runs of $\eta^{\delta,\zeta}(\ell) \neq 0$ followed by phase changes in the intervals $[a_i, b_i)$, for $i = 1, \dots \bar{N}$, see (4.12). So to estimate the ratio of the partition function in (4.19) and (4.20), it is convenient to separate the contribution given by those intervals in which the spin configurations undergo to a phase change, *i.e* in which the block spin variables are $\eta^{\delta,\zeta}(\ell) = 0$, from those intervals in which the block spin variables are $\eta^{\delta,\zeta}(\ell) \neq 0$. This can be achieved *cutting* at suitable points the partition function. We require these points to be such that $\eta^{\delta,\zeta_5}(\ell) \neq 0$ to obtain error terms which are negligible. We start proving an upper bound for (4.20). To facilitate the reading, as before, we use indexes with $'$ and $''$ to denote the points ℓ where $\eta^{\delta,\zeta_5}(\ell) \neq 0$. Denote a generic term in (4.20) by

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}) \equiv \frac{Z^{0,0}_{[n'_0,n'_{\bar{N}+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar{N}+1}]}(m),\ \eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m^*),\ \eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m^*)\Big)}{Z^{0,0}_{[n'_0,n'_{\bar{N}+1}]}\Big(\mathcal{P}^{\rho}_{[n'_0,n'_{\bar{N}+1}]}(m^*),\ \eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m^*),\ \eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m^*)\Big)}. \tag{4.43}$$

We have the following:

Lemma 4.11 *Under the same hypothesis of Proposition 4.2, on the probability space $\Omega_1 \setminus (\Omega_3 \cup \Omega_4)$, and for ζ_5 as in (4.18), we have*

$$\begin{aligned} \mathcal{Z}(n'_0, n'_{\bar{N}+1}) \le\ & e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{-L\le r\le L}\left[|D\tilde{u}(r)|-|D\tilde{u}^*_\gamma(r)|\right]} e^{\frac{\beta}{\gamma}\sum_{i=1}^{\bar{N}}\frac{\tilde{u}(r_i)-\tilde{u}^*_\gamma(r_i)}{2\tilde{m}_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]} \\ & \times e^{\frac{\beta}{\gamma}\bar{N}\left[4\zeta_5+8\delta^*+\gamma\log\frac{\rho}{\gamma}+\gamma\log L_1+\frac{20V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/4(2+a)}}+32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}}\right]} \\ & + \bar{N}^2 e^{\bar{N}\log\frac{\rho}{\gamma}} e^{\frac{\beta}{\gamma}(8\delta^*+4\zeta)} e^{-\frac{\beta}{\gamma}L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}. \end{aligned} \tag{4.44}$$

Proof: Recalling(2.33), and (4.12), one sees that in each interval $[a_i, b_i]$, there is a single phase change on a length R_2 for m or m^*. There are three possible cases:
Case 1 $[a_i, b_i] \in C_\gamma(u)$ and $[a_i, b_i] \in B_\gamma(u^*_\gamma)$. Therefore

$$\begin{aligned} \eta(a_i, m) &= -\eta(b_i, m) \neq 0 \\ \eta(a_i, m^*) &= \eta(b_i, m^*) \neq 0. \end{aligned} \tag{4.45}$$

Case 2 $[a_i, b_i] \in B_\gamma(u)$ and $[a_i, b_i] \in C_\gamma(u^*_\gamma)$. Therefore

$$\begin{aligned} \eta(a_i, m) &= \eta(b_i, m) \neq 0 \\ \eta(a_i, m^*) &= -\eta(b_i, m^*) \neq 0. \end{aligned} \tag{4.46}$$

Case 3 $[a_i, b_i] \in C_\gamma(u)$ and $[a_i, b_i] \in C_\gamma(u^*_\gamma)$. Therefore

$$\begin{aligned} \eta(a_i, m) &= -\eta(b_i, m) \neq 0 \\ \eta(a_i, m^*) &= -\eta(b_i, m^*) \neq 0. \end{aligned} \tag{4.47}$$

In the first two cases there exists an unique $x_i \in [a_i, b_i]$, see (4.13), so that, in the the case 1, $|D\tilde{m}(x_i))| > 0$ and in the case 2, $|D\tilde{m}^*(x_i)| > 0$. In the case 3, both m and m^* have one jump in $[a_i, b_i]$. Recalling (4.15) and Definition 2.1 we denote

$$\mathcal{W}_1(\ell_i, m) \equiv \mathcal{W}_1([\ell_i - R_2, \ell_i + R_2], R_2, \zeta) \cap \{\eta^{\delta,\zeta}(\ell_i - R_2) = \eta(a_i, m), \eta^{\delta,\zeta}(\ell_i + R_2) = \eta(b_i, m)\}, \tag{4.48}$$

the set of configurations undergoing to a phase change induced by m in $[\ell_i - R_2, \ell_i + R_2]$. We denote in the cases 1 and 3

$$\mathcal{P}^{\rho}_{[a_i,b_i]}(m, \ell_i, i) \equiv \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [a_i, \ell_i - R_2 - 1]) \cap \mathcal{W}_1(\ell_1, m) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, b_i]) \tag{4.49}$$

and in the case 2

$$\begin{aligned}&\mathcal{P}^{\rho}_{[a_i,b_i]}(m, \ell_i, i) \equiv \\ &\mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [a_i, \ell_i - R_2 - 1]) \cap \mathcal{R}^{\delta,\zeta}(\eta(a_i, m), [\ell_i - R_2, \ell_i + R_2]) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i, m), [\ell_i + R_2 + 1, b_i]).\end{aligned} \tag{4.50}$$

The set $\mathcal{P}^{\rho}_{[a_i,b_i]}(m, \ell_i, i)$ denotes the spin configurations which, in the case 1 and 3, have a jump in the interval $[a_i, b_i]$, starting after the point $\ell_i - R_2$ and ending before $\ell_i + R_2$ and close to different equilibrium values in $[a_i, b_i] \setminus [\ell_i - R_2, \ell_i + R_2]$. In the case 2, it denotes the spin configurations which are in all $[a_i, b_i]$ close to one equilibrium value, namely they do not have jumps. The ℓ_i in this last case is written for future use. We use for both m and m^* the notation (4.49) and (4.50). In the case 3 both m and m^* have a jump in $[a_i, b_i]$. Obviously we have

$$\mathcal{P}^{\rho}_{[a_i,b_i]}(m) \subset \bigcup_{\ell_i \in [a_i+R_2+1, b_i-R_2-1]} \mathcal{P}^{\rho}_{[a_i,b_i]}(m, \ell_i, i). \tag{4.51}$$

To get an upper bound for (4.43), we use the subadditivity of the numerator in (4.43) to treat the $\cup$ in (4.51) obtaining a sum over $\ell_i \in [a_i + R_2 + 1, b_i - R_2 - 1]$. For the denominator we obtain an upper bound simply restricting to the subset of configurations which is suitable for us, namely

$$\mathcal{P}^{\rho}_{[a_i,b_i]}(m^*) \supset \mathcal{P}^{\rho}_{[a_i,b_i]}(m^*, \ell_i, i). \tag{4.52}$$

To short notation, let $\underline{\ell} \subset [\underline{a}, \underline{b}] \equiv \{\ell_i \in [a_i + R_2 + 1, b_i - R_2 - 1], \forall i, 1 \le i \le \bar{N}\}$ and set

$$\mathcal{A}(m, \underline{\ell}) \equiv \Big(\mathcal{P}^{\rho}_{[n'_0, n'_{\bar{N}+1}]}(m) \cap_{i=1}^{\bar{N}} \mathcal{P}^{\rho}_{[a_i,b_i]}(m, \ell_i, i),\ \eta^{\delta,\zeta_5}(n'_0) = \eta(n'_0, m^*),\ \eta^{\delta,\zeta_5}(n'_{\bar{N}+1}) = \eta(n'_{\bar{N}+1}, m^*)\Big), \tag{4.53}$$

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}) \equiv \frac{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m, \underline{\ell})\Big)}{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m^*, \underline{\ell})\Big)}. \tag{4.54}$$

Therefore, recalling (4.43), we can write

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}) \le \sum_{\underline{\ell} \subset [\underline{a}, \underline{b}]} \mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}). \tag{4.55}$$

The number of terms in the sum in (4.55) does not exceed $\prod_{i=i}^{\bar{N}} (b_i - a_i) \le \exp(\bar{N} \log(\rho/\gamma))$. For future use, when B is an event let us define

$$\mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}; B) \equiv \frac{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m, \underline{\ell}) \cap B\Big)}{Z^{0,0}_{[n'_0, n'_{\bar{N}+1}]}\Big(\mathcal{A}(m^*, \underline{\ell})\Big)}. \tag{4.56}$$

For ℓ_0 defined in (4.21), for the very same L_1 to be chosen later and ζ_5 that satisfies (4.18) , let us denote $\bar{R}_2 = R_2 + \ell_0$ and define

$$\begin{aligned}&\mathcal{D}(m, \underline{\ell}) \equiv \\ &\quad \cup_{1 \le i \le \bar{N}} \left(\mathcal{R}^{\delta,\zeta}(\eta(\ell_i - R_2, m), [\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right) \cup \\ &\quad \cup_{1 \le i \le \bar{N}} \left(\mathcal{R}^{\delta,\zeta}(\eta(\ell_i + R_2, m), [\ell_i + R_2, \ell_i + R_2 + 2\ell_0 + L_1]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])\right).\end{aligned} \tag{4.57}$$

The $\mathcal{D}(m,\underline{\ell})$ is the set of configurations which are simultaneously ζ close and ζ_5 distant, (recall $\zeta > \zeta_5$), from the equilibrium values in the interval $[\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2] \cup [\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1]$ where ℓ_i are chosen as in (4.52). Recalling (4.56) and Proposition 4.7 one gets

$$\sum_{\underline{\ell}\subset[\underline{a},\underline{b}]} \mathcal{Z}(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \mathcal{D}(m,\underline{\ell})) \le \bar{N}^2 e^{\bar{N}\log\frac{\rho}{\gamma}} e^{\frac{\beta}{\gamma}(8\delta^*+4\zeta)} e^{-\frac{\beta}{\gamma}L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3}. \tag{4.58}$$

To get (4.58) one cuts at the points $\ell_i + R_2$ and $\ell_i + R_2 + 2\ell_0 + L_1$ for the set $\mathcal{R}^{\delta,\zeta}(\eta(\ell_i + R_2, m), [\ell_i + R_2, \ell_i + R_2 + 2\ell_0 + L_1]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])$, and at the points $\ell_i - R_2 - L_1 - 2\ell_0$ and $\ell_i - R_2$ for the set $\mathcal{R}^{\delta,\zeta}(\eta(\ell_i - R_2, m), [\ell_i - R_2 - L_1 - 2\ell_0, \ell_i - R_2]) \cap \mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])$. Notice that we cut at points $\eta^{\delta,\zeta} \neq 0$ and each time we make the error $e^{\frac{\beta}{\gamma}(2\zeta+4\delta^*)}$. This is the only place where making an error so large does not cause a problem. Namely we can choose L_1 suitable in (4.58) so that $L_1\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3 > (8\delta^* + 4\zeta)$. Furthermore denote

$$\mathcal{B}(\underline{\ell}) \equiv \cap_{1\le i\le \bar{N}} \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right)^c \cap \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1])\right)^c. \tag{4.59}$$

Since for each $\underline{\ell}$, $\mathcal{A}(m,\underline{\ell}) \cap \mathcal{D}(m,\underline{\ell})^c \subset \mathcal{A}(m,\underline{\ell}) \cap \mathcal{B}(\underline{\ell})$ we are left to estimate

$$\sum_{\underline{\ell}\subset[\underline{a},\underline{b}]} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \mathcal{B}(\underline{\ell})\right).$$

On each $\mathcal{A}(m,\underline{\ell}) \cap \left(\mathcal{O}^{\delta,\zeta_5}([\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2])\right)^c$, $1 \le i \le \bar{N}$, there exists at least one block, say $[n'_i, n'_i+1)$ contained in $[\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2)$ with $\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i, m)$. Making the same on the right of ℓ_i and indexing n''_i the corresponding block where $\eta^{\delta,\zeta_5}(n''_i) = \eta(b_i, m)$, one gets

$$\begin{aligned} &\sum_{\underline{\ell}\subset[\underline{a},\underline{b}]} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}, \mathcal{B}(\underline{\ell})\right) \le \\ &\sum_{\underline{\ell}\subset[\underline{a},\underline{b}]} \sum_{\substack{\underline{n}'\subset[\underline{\ell}-\bar{R}_2-L_1,\underline{\ell}-\bar{R}_2] \\ \underline{n}''\subset[\underline{\ell}+\bar{R}_2,\underline{\ell}+\bar{R}_2+L_1]}} \mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \cap_{1\le i\le\bar{N}}\{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i,m), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i,m)\}\right). \end{aligned} \tag{4.60}$$

The number of terms in the second sum of (4.60) does not exceed $\exp(2\bar{N}(\log L_1))$. Consider now a generic term in (4.60),

$$\mathcal{Z}\left(n'_0, n'_{\bar{N}+1}, \underline{\ell}; \cap_{1\le i\le\bar{N}}\{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i,m), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i,m)\}\right). \tag{4.61}$$

Recalling (4.56), we cut the numerator of the partition function, as in (4.39), at the points $\underline{n}'$ and $\underline{n}''$ to get an upper bound. Each time we cut we get the error term $e^{\frac{\beta}{\gamma}(2\delta^*+\zeta_5)}$. In the denominator, see (4.54), restrict the configurations to be in

$$\mathcal{A}(m^*,\underline{\ell}) \cap_{1\le i\le\bar{N}} \left\{\eta^{\delta,\zeta_5}(n'_i) = \eta(a_i,m^*), \eta^{\delta,\zeta_5}(n''_i) = \eta(b_i,m^*)\right\} \tag{4.62}$$

and then cut at all the points $\underline{n}'$ and $\underline{n}''$. In this way we obtain an upper bound for (4.61). We use notation (4.49) (case 1 and 3) and (4.50) (case 2) after cutting at n'_i and n''_i. Note that $\eta(n'_i+1) = \eta(a_i,m)$ and $\eta(n''_i-1) = \eta(b_i,m)$ therefore we have in the case 1 and 3, see (4.49),

$$\begin{aligned} &\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i) = \\ &\quad \mathcal{R}^{\delta,\zeta}(\eta(a_i,m),[n'_i+1,\ell_i-R_2-1]) \cap \mathcal{W}_1(\ell_i,m) \cap \mathcal{R}^{\delta,\zeta}(\eta(b_i,m),[\ell_i+R_2+1,n''_i-1]), \end{aligned} \tag{4.63}$$

in the case 2, see (4.50),

$$\begin{aligned}&\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)=\mathcal{R}^{\delta,\zeta}(\eta(a_i,m),[n'_i+1,\ell_i-R_2-1])\cap\\&\quad\mathcal{R}^{\delta,\zeta}(\eta(a_i,m),[\ell_i-R_2,\ell_i+R_2])\cap\mathcal{R}^{\delta,\zeta}(\eta(b_i,m),[\ell_i+R_2+1,n''_i-1]).\end{aligned}\tag{4.64}$$

For the remaining parts corresponding to runs between two phase changes, *i.e* the intervals $[n''_i,n'_{i+1}]$, $n''_i\in[a_i,b_i]$ and $n'_{i+1}\in[a_{i+1},b_{i+1}]$, for $i\in\{1,\dots,\bar N\}$, we denote

$$\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m,\zeta_5)\equiv\mathcal{R}^{\delta,\zeta}(\eta(b_i,m),[n''_i+1,n'_{i+1}-1])\cap\{\eta^{\delta,\zeta_5}(n''_i)=\eta^{\delta,\zeta_5}(n'_{i+1})=\eta(b_i,m)\}.\tag{4.65}$$

Similarly in the intervals $[n'_0,n'_1]$, and $[n''_{\bar N},n'_{\bar N+1}]$, recalling (4.22) and (4.23), we have

$$\begin{aligned}&\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5)\equiv\mathcal{R}^{\delta,\zeta}(\eta(v_1,m),[n'_0,n'_1])\cap\{\eta^{\delta,\zeta_5}(n'_0)=\eta^{\delta,\zeta_5}(n'_1)=\eta(v_1,m)\}\\&=\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5),\end{aligned}\tag{4.66}$$

$$\begin{aligned}&\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5)\equiv\mathcal{R}^{\delta,\zeta}(\eta(v_2,m),[n''_{\bar N},n'_{\bar N+1}])\cap\{\eta^{\delta,\zeta_5}(n''_{\bar N})=\eta^{\delta,\zeta_5}(n'_{\bar N+1})=\eta(v_2,m)\}\\&=\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5).\end{aligned}\tag{4.67}$$

As a result we have

$$\begin{aligned}&\mathcal{Z}\left(n'_0,n'_{\bar N+1},\underline{\ell};\cap_{1\le i\le\bar N}\{\eta^{\delta,\zeta_5}(n'_i)=\eta(a_i,m),\eta^{\delta,\zeta_5}(n''_i)=\eta(b_i,m)\}\right)\le\\&e^{+\bar N\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\frac{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5)\right)}{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5)\right)}\times\\&\prod_{i=1}^{\bar N-1}\left(\frac{Z^{m,m}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n'_i+1,n''_i-1]}\left(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m^*,\ell_i,i)\right)}\frac{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i,n'_{i+1}]}(m^*,\zeta_5)\right)}\right)\times\\&\frac{Z^{m,m}_{[n'_{\bar N}+1,n''_{\bar N}-1]}\left(\mathcal{P}^{\rho}_{[n'_{\bar N}+1,n''_{\bar N}-1]}(m,\ell_{\bar N},\bar N)\right)}{Z^{m^*,m^*}_{[n'_{\bar N}+1,n''_{\bar N}-1]}\left(\mathcal{P}^{\rho}_{[n'_{\bar N}+1,n''_{\bar N}-1]}(m^*,\ell_{\bar N},\bar N)\right)}\frac{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5)\right)}.\end{aligned}\tag{4.68}$$

Remark 4.12 . Note that the boundary conditions of restricted partition functions as

$$Z^{m,m}_{[n'_i+1,n''_i-1]}(\mathcal{P}^{\rho}_{[n'_i+1,n''_i-1]}(m,\ell_i,i))$$

in (4.68) are related on the left to $\eta(a_i,m)$ and on the right to $\eta(b_i,m)$, see (4.63) and (4.64).

Now the goal is to estimate separately all the ratios in the right hand side of (4.68). It follows from (4.22), (4.66), and (4.67) that

$$\frac{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m,\zeta_5)\right)}{Z^{0,0}_{[n'_0,n'_1]}\left(\mathcal{P}^{\rho}_{[n'_0,n'_1]}(m^*,\zeta_5)\right)}=\frac{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_{\bar N},n'_{\bar N+1}]}\left(\mathcal{P}^{\rho}_{[n''_{\bar N},n'_{\bar N+1}]}(m^*,\zeta_5)\right)}=1.$$

The remaining ratios are estimated in Lemma 4.13, Corollary 4.14 and Lemma 4.15 given below.

Collecting We insert the results of Lemma 4.13, Corollary 4.14 and Lemma 4.15 in (4.68). To write in a unifying way the contributions of the jumps we note that for (4.84)

$$-\mathcal{F}^* = -\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{a_i\le s\le b_i}|D\tilde{m}(s)| = -\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{a_i\le s\le b_i}\left(|D\tilde{m}(s)|-|D\tilde{m}^*(s)|\right) \tag{4.69}$$

since in the case 1, see (4.45), $\sum_{a_i\le s\le b_i}|D\tilde{m}^*(s)|=0$. For (4.85)

$$+\mathcal{F}^* = \frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{a_i\le s\le b_i}|D\tilde{m}^*(s)| = -\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{a_i\le s\le b_i}\left(|D\tilde{m}(s)|-|D\tilde{m}^*(s)|\right) \tag{4.70}$$

since in the case 2, see (4.46), $\sum_{a_i\le s\le b_i}|D\tilde{m}(s)|=0$. Moreover, since neither $\tilde{m}$ nor $\tilde{m}^*$ have jump in $[b_i+1,a_{i+1}]$ for $i\in\{1,\dots,\bar{N}\}$, in $[v_1,a_1-1]$, and in $[b_{\bar{N}}+1,v_2]$, one gets simply

$$\prod_{i=1}^{\bar{N}} e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{a_i\le s\le b_i}[|D\tilde{m}(s)|-|D\tilde{m}^*(s)|]} = e^{-\frac{\beta}{\gamma}\frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{-L\le r\le L}\left[|D\tilde{u}(r)|-|D\tilde{u}^*_\gamma(r)|\right]}. \tag{4.71}$$

Using (4.78), the random terms give a contribution

$$e^{\frac{\beta}{\gamma}\sum_{i=1}^{\bar{N}}\frac{\tilde{u}(r_i)-\tilde{u}^*_\gamma(r_i)}{2\tilde{m}_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]}. \tag{4.72}$$

It remains to collect the error terms, see (4.58), (4.60), (4.68), (4.78), and Lemma 4.15. Denote

$$\mathcal{E}_1 \equiv \bar{N}\left[4\zeta_5+8\delta^*+\gamma\log\frac{\rho}{\gamma}+\gamma\log L_1+\frac{20V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/4(2+a)}}+32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}}\right], \tag{4.73}$$

$$-\mathcal{A}_2 \equiv 2\frac{\gamma}{\beta}\log\bar{N}+\frac{\gamma}{\beta}\bar{N}\log\frac{\rho}{\gamma}+8\delta^*+4\zeta-L_1\frac{\kappa(\beta,\theta)}{8}\delta\zeta_5^3, \tag{4.74}$$

and

$$\mathcal{A} \equiv \frac{\mathcal{F}^*}{2\tilde{m}_\beta}\sum_{-L\le r\le L}\left[|D\tilde{u}(r)|-|D\tilde{u}^*_\gamma(r)|\right]-\sum_{i=1}^{\bar{N}}\frac{\tilde{u}(r_i)-\tilde{u}^*_\gamma(r_i)}{2\tilde{m}_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]. \tag{4.75}$$

We have proved

$$\mathcal{Z}(n'_0,n'_{\bar{N}+1}) \le e^{-\frac{\beta}{\gamma}\mathcal{A}}e^{\frac{\beta}{\gamma}\mathcal{E}_1}+e^{-\frac{\beta}{\gamma}\mathcal{A}_2} \tag{4.76}$$

that entails (4.44). ■

Next we state the lemmas used for estimating the different ratios in (4.68).

Lemma 4.13 *Under the same hypothesis of Proposition 4.2 and on the probability space $\Omega_1\setminus\Omega_3$, for all $1\le i\le\bar{N}-1$, for all n''_i,n'_i, we have*

$$\frac{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^\rho_{[n''_i,n'_{i+1}]}(m,\zeta_5)\right)}{Z^{0,0}_{[n''_i,n'_{i+1}]}\left(\mathcal{P}^\rho_{[n''_i,n'_{i+1}]}(m^*,\zeta_5)\right)} = \begin{cases} 1 & \text{when } \eta(b_i,m)=\eta(b_i,m^*); \\ e^{\pm\frac{\beta}{\gamma}\frac{1}{4c^2(\beta,\theta)g(\delta^*/\gamma)}}e^{\frac{\beta}{\gamma}\frac{\tilde{m}(b_i)-\tilde{m}^*(b_i)}{2\tilde{m}_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in\gamma[n''_i+1,n'_{i+1}-1]}\chi(\alpha)\right]} & \\ \qquad \text{when} \quad \eta(b_i,m)=-\eta(b_i,m^*), & \end{cases} \tag{4.77}$$

where in the last term we have an upper bound for $\pm = +$ and a lower bound for $\pm = -$.

Proof: When $\eta(b_i, m) = \eta(b_i, m^*)$ (4.77) is immediate, see definition (4.65). When $\eta(b_i, m) = -\eta(b_i, m^*)$ the estimate is a direct consequence of Lemma 4.19. ■

The r.h.s of (4.77) gives when $\eta(b_i, m) = -\eta(b_i, m^*)$ a term which should give the second sum in the right hand side of (4.11). However in (4.77) one has $\sum_{\alpha:\, \epsilon\alpha\in\gamma[n''_i+1, n'_{i+1}-1]} \chi(\alpha)$ instead of $\sum_{\alpha:\, \epsilon\alpha\in[r_i, r_{i+1})} \chi(\alpha)$ in (4.11). To obtain the term in (4.11) we add the missed random field to reconstruct $\sum_{\alpha:\, \epsilon\alpha\in[r_i, r_{i+1})} \chi(\alpha)$. We therefore need to subtract the same term we added as a result one has

Corollary 4.14 *Under the same hypothesis of Proposition 4.2 and on the probability space $\Omega_1 \setminus (\Omega_3 \cup \Omega_4)$, with $I\!P(\Omega_4) \le e^{-(\log g(\delta^*/\gamma))\left(1-\frac{1}{\log\log g(\delta^*/\gamma)}\right)}$, for all $1 \le i \le \bar N - 1$, for all n''_i, n'_i, when $\eta(b_i, m) = -\eta(b_i, m^*)$ we have*

$$\frac{Z^{0,0}_{[n''_i, n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i, n'_{i+1}]}(m, \zeta_5)\right)}{Z^{0,0}_{[n''_i, n'_{i+1}]}\left(\mathcal{P}^{\rho}_{[n''_i, n'_{i+1}]}(m^*, \zeta_5)\right)} = e^{\pm\frac{\beta}{\gamma}\frac{20V(\beta,\theta)}{(g(\delta^*/\gamma))^{1/4(2+a)}}} e^{\frac{\beta}{\gamma}\frac{\tilde u(r_i) - \tilde u^*_\gamma(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\, \epsilon\alpha\in[r_i, r_{i+1})}\chi(\alpha)\right]} \tag{4.78}$$

where we have an upper bound for $\pm = +$ and a lower bound for $\pm = -$.

Proof: Recalling that $\gamma x_i = r_i$, see (4.13), we need to estimate

$$\mathcal{K}(i, n''_i, n'_{i+1}) = \sum_{\alpha:\, \epsilon\alpha\in\gamma[x_i, n''_i]} \chi(\alpha) + \sum_{\alpha:\, \epsilon\alpha\in\gamma[n'_{i+1}, x_{i+1}]} \chi(\alpha) \tag{4.79}$$

uniformly with respect to $1 \le i \le \bar N$, x_i, n''_i, n'_{i+1}. It is enough to estimate

$$\max_{1\le i\le \bar N} \max_{x_i} \max_{1\le \ell\le L_1+\rho\gamma^{-1}} \Big| \sum_{\alpha:\epsilon\alpha\in\gamma[x_i, \ell+x_i]} \chi(\alpha)\Big|. \tag{4.80}$$

However since the point x_i might be random and depending on $\chi(\alpha)$, a little care is needed. An upper bound for (4.80) is clearly

$$\mathcal{K}(Q, L_1, \rho, \epsilon) \equiv \max_{\alpha_0:\epsilon\alpha_0\in[-Q,+Q]} \max_{1\le\epsilon\bar\alpha\le\gamma L_1+\rho} \Big|\sum_{\alpha=\alpha_0}^{\bar\alpha} \chi(\alpha)\Big|. \tag{4.81}$$

Using, Levy inequality, (3.57) and exponential Markov inequality, one has

$$\begin{aligned} &I\!P\left[\mathcal{K}(Q, L_1, \rho, \epsilon) \ge 2V(\beta,\theta)\sqrt{2(\gamma L_1+\rho)\log(g^5(\frac{\delta^*}{\gamma}))}\right] \\ &\le \frac{2Q+1}{\epsilon} I\!P\left[\max_{1\le\epsilon\bar\alpha\le\gamma L_1+\rho}\Big|\sum_{\alpha=\alpha_0}^{\bar\alpha}\chi(\alpha)\Big| \ge 2V(\beta,\theta)\sqrt{2(\gamma L_1+\rho)\log(g^5(\frac{\delta^*}{\gamma}))}\right] \\ &\le \frac{2Q+1}{\epsilon}\frac{1}{g^5(\frac{\delta^*}{\gamma})} \le e^{-\log g(\frac{\delta^*}{\gamma})\left(1-\frac{1}{\log\log g(\frac{\delta^*}{\gamma})}\right)}, \end{aligned} \tag{4.82}$$

where we have used (2.65) and (2.67) at the last step. Recalling that $0 \le L_1 + \ell_0 \le \rho\gamma^{-1}$ and (2.64), one has

$$2(\gamma L_1 + \rho) \log(g^5(\frac{\delta^*}{\gamma})) \le 4\rho \log(g^5(\frac{\delta^*}{\gamma})) \le 4 \left(\frac{5}{g(\frac{\delta^*}{\gamma})} \right)^{1/(2+a)} \log(g^5(\frac{\delta^*}{\gamma})) \le 5^2 \left(\frac{1}{g(\frac{\delta^*}{\gamma})} \right)^{1/2(2+a)} . \quad (4.83)$$

that entails (4.78) after an easy computation. ■

Next we estimate the remaining type of ratio in (4.68). Recall that $n_i' \in [\ell_i - \bar{R}_2 - L_1, \ell_i - \bar{R}_2]$ and $n_i'' \in [\ell_i + \bar{R}_2, \ell_i + \bar{R}_2 + L_1]$ with $\bar{R}_2 = R_2 + \ell_0$, where ℓ_0 is defined in (4.34).

Lemma 4.15 *On $\Omega_1 \setminus \Omega_3$, choosing the parameters as in Subsection 2.5, for all $1 \le i \le \bar{N}$, in the case 1, we have*

$$\frac{Z^{m,m}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m^*,\ell_i,i)\right)} = e^{-\frac{\beta}{\gamma}(\mathcal{F}^* \pm 32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \quad (4.84)$$

In the case 2, we have

$$\frac{Z^{m,m}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m^*,\ell_i,i)\right)} = e^{+\frac{\beta}{\gamma}(\mathcal{F}^* \pm 32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \quad (4.85)$$

In the case 3, we have

$$\frac{Z^{m,m}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m,\ell_i,i)\right)}{Z^{m^*,m^*}_{[n_i'+1,n_i''-1]}\left(\mathcal{P}^{\rho}_{[n_i'+1,n_i''-1]}(m^*,\ell_i,i)\right)} = e^{\pm\frac{\beta}{\gamma}(32\theta(R_2+\ell_0+L_1)\sqrt{\frac{\gamma}{\delta^*}})}. \quad (4.86)$$

Remark 4.16 . Note that here one needs to have $L_1\sqrt{\frac{\gamma}{\delta^*}} \downarrow 0$.

The proof of (4.84) and (4.85) follows from Lemma 4.18. The (4.86) is a consequence of (4.84) and (4.85). Next we estimate from below the r.h.s. of (4.19).

Lemma 4.17 *Under the same hypothesis of Proposition 4.2 and on the probability space $\Omega_1 \setminus (\Omega_3 \cup \Omega_4)$, for ζ_5 as in (4.18), we have*

$$\begin{aligned} &\frac{Z^{0,0}_{[v_1-1,v_2+1]}\left(\mathcal{P}^{\rho}_{[v_1,v_2]}(m), \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*), \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\right)}{Z^{0,0}_{[v_1-1,v_2+1]}\left(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*), \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*), \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\right)} \\ &\ge \left(e^{\frac{\beta}{\gamma}(\mathcal{A}+\mathcal{E}_1)} + e^{-\frac{\beta}{\gamma}\mathcal{A}_2}\right)^{-1} \end{aligned} \quad (4.87)$$

where $\mathcal{A}$, $\mathcal{E}_1$, and $\mathcal{A}_2$ are defined in (4.75), (4.73), and (4.74) respectively.

Proof: Obviously one can get the lower bound simply proving an upper bound for the inverse of l.h.s. of (4.87), i.e.

$$\frac{Z^{0,0}_{[v_1-1,v_2+1]}\left(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\, \eta^{\delta,\zeta_5}(v_1-1) = \eta([v_1-1,m^*),\, \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\right)}{Z^{0,0}_{[v_1-1,v_2+1]}\left(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\, \eta^{\delta,\zeta_5}(v_1-1) = \eta(v_1-1,m^*),\, \eta^{\delta,\zeta_5}(v_2+1) = \eta(v_2+1,m^*)\right)}. \quad (4.88)$$

Note that $\eta(v_1-1,m^*)=\eta(v_1-1,m)$ and $\eta(v_2+1,m^*)=\eta(v_2+1,m)$ and in the proof of the upper bound, see Lemma 4.11, we never used that m^* in the denominator is the one given in Theorem 2.3. Then (4.88) is equal to

$$\frac{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m^*),\, \eta^{\delta,\zeta_5}(v_1-1)=\eta([v_1-1,m),\, \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m)\Big)}{Z^{0,0}_{[v_1-1,v_2+1]}\Big(\mathcal{P}^{\rho}_{[v_1,v_2]}(m),\, \eta^{\delta,\zeta_5}(v_1-1)=\eta(v_1-1,m),\, \eta^{\delta,\zeta_5}(v_2+1)=\eta(v_2+1,m)\Big)}. \tag{4.89}$$

Then by Lemma 4.11 we obtain (4.87). ∎

Proof of Proposition 4.2 To prove (4.11), we use Lemma 4.9, then Lemma 4.17 to get a lower bound and Lemma 4.11 and Corollary 4.14 to get an upper bound. For the lower bound we get applying (4.19) and (4.87)

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big)\geq e^{-\frac{\beta}{\gamma}(4\zeta_5+8\delta^*)}\Big(1-2K(Q)e^{-\frac{\beta}{\gamma}\frac{1}{g(\delta^*/\gamma)}}-2e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3}\Big)\Big(e^{\frac{\beta}{\gamma}\mathcal{A}}e^{\frac{\beta}{\gamma}\mathcal{E}_1}+e^{-\frac{\beta}{\gamma}\mathcal{A}_2}\Big)^{-1}. \tag{4.90}$$

For the upper bound we get

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big)\leq e^{-\frac{\beta}{\gamma}\mathcal{A}}e^{+\frac{\beta}{\gamma}\mathcal{E}_1}+2e^{-\frac{\beta}{\gamma}\mathcal{A}_2} \tag{4.91}$$

where $\mathcal{A}_2$ is defined in (4.74). To get (4.11) from (4.91), one needs $\mathcal{A}_2>\mathcal{A}$, this will be a consequence of an upper bound on $\mathcal{A}$ and a lower bound on $\mathcal{A}_2$. We start estimating the terms of $\mathcal{A}$. We easily obtain

$$\frac{\mathcal{F}^*}{4\tilde m_\beta}\sum_{-L\leq r\leq L}\big[\|Du(r)\|_1-\|Du^*_\gamma(r)\|_1\big]\leq \mathcal{F}^*\left[N_{[-L,L]}(u)+N_{[-Q,Q]}(u^*_\gamma)\right]. \tag{4.92}$$

We use that $N_{[-Q,Q]}(u^*_\gamma)\leq K(Q)$, see (5.91), where $K(Q)$ is given in (2.35). If L is finite for all γ, then $N_{[-L,L]}(u)$ is bounded since $u\in BV_{loc}$. When L diverges as Q when $\gamma\downarrow 0$ from the assumption (4.10) we have that

$$\bar N\leq N_{[-L,L]}(u)+N_{[-Q,Q]}(u^*_\gamma)\leq [f(Q)+1]K(Q) \tag{4.93}$$

where we set

$$f(Q)=e^{(\frac{1}{8+4a}-b)(\log Q)(\log\log Q)}. \tag{4.94}$$

The second term of $\mathcal{A}$ can be estimated as

$$\begin{aligned}\left|\sum_{i=1}^{\bar N}\frac{\tilde u(r_i)-\tilde u^*_\gamma(r_i)}{2\tilde m_\beta}\left[\sum_{\alpha:\,\epsilon\alpha\in[r_i,r_{i+1})}\chi(\alpha)\right]\right| &\leq \bar N\max_{\{-\frac{Q}{\epsilon}\leq\alpha_0\leq\frac{Q}{\epsilon}\}}\max_{\{\alpha_0\leq\bar\alpha\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=\alpha_0}^{\bar\alpha}\chi(\alpha)\right|\\ &\leq 2\bar N\max_{\{-\frac{Q}{\epsilon}\leq\bar\alpha\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=-\frac{Q}{\epsilon}}^{\bar\alpha}\chi(\alpha)\right|.\end{aligned} \tag{4.95}$$

To estimate the last term in (4.95), we use Levy inequality, (3.57) and exponential Markov inequality to get

$$I\!P\Big[\max_{\{-\frac{Q}{\epsilon}\leq\bar\alpha\leq\frac{Q}{\epsilon}\}}\left|\sum_{\alpha=-\frac{Q}{\epsilon}}^{\bar\alpha}\chi(\alpha)\right|\geq\sqrt{3}V(\beta,\theta)\sqrt{[2Q+1]\log(g(\frac{\delta^*}{\gamma}))}\Big]\leq 4e^{-\log(g(\frac{\delta^*}{\gamma}))}=\frac{4}{g(\frac{\delta^*}{\gamma})}. \tag{4.96}$$

Denote Ω_5 the probability space for which (4.96) holds. Then for $\omega \in \Omega_1 \setminus (\Omega_3 \cup \Omega_4 \cup \Omega_5)$ and γ_0 small enough, one has

$$\mathcal{A} \leq 2[f(Q)+1]K(Q)V(\beta,\theta)\sqrt{(2Q+1)\log(g(\frac{\delta^*}{\gamma}))} \leq \bar{c}(\beta,\theta)f(Q)Q^3 \tag{4.97}$$

for some $\bar{c}(\beta,\theta)$. The last inequality in (4.97) is obtained from the choice of $f(Q)$ in (4.94), the one of $K(Q)$ in (2.35) and the choice of Q in (2.67). Namely, from (2.67) $Q^2 g(\delta^*/\gamma) \leq g^2(\delta^*/\gamma)$. Notice that in $\mathcal{A}_2$, see (4.74), L_1 enters. We make the following choice of L_1

$$L_1 = \left(g(\frac{\delta^*}{\gamma})\right)^{19/2}. \tag{4.98}$$

This choice satisfies the requirement in Proposition 4.9, i.e. $L_1 < \frac{\rho}{\gamma}$, see (2.64). Furthermore as in [16] we make the choice

$$\zeta_5 = \frac{1}{2^{18}c^6(\beta,\theta)}\frac{1}{g^3(\delta^*/\gamma)} \tag{4.99}$$

for some constant $c(\beta,\theta)$. Obviously (4.99) satisfies requirement (4.18) provided ζ is chosen according (2.62). Since $Q = g(\delta^*/\gamma)^{\frac{1}{\log\log g(\delta^*/\gamma)}}$, see (2.67), we have $\log g(\delta^*/\gamma) = (\log Q)(\log\log g(\delta^*/\gamma))$. Iterating this equation, for γ_0 small enough to have $\log\log\log g(\delta^*/\gamma) > 0$, one gets

$$\log g(\delta^*/\gamma) = (\log Q)(\log\log Q)\Big(1 + \frac{\log\log\log g(\delta^*/\gamma)}{\log\log g(\delta^*/\gamma) - \log\log\log g(\delta^*/\gamma)}\Big) \geq (\log Q)(\log\log Q). \tag{4.100}$$

Therefore, recalling (2.63) and using (4.94) one can check that

$$L_1\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3 \geq c(\beta,\theta)f(Q)Q^3. \tag{4.101}$$

It is not difficult to check that (4.101) implies

$$L_1\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3 > 2\gamma\log\bar{N} + \gamma\bar{N}\log\frac{\rho}{\gamma} + 8\delta^* + 4\zeta. \tag{4.102}$$

Therefore, recalling (4.74), (4.101) entails $\mathcal{A}_2 > \mathcal{A}$ and finally one gets

$$\mu_{\beta,\theta,\gamma}\big(\mathcal{P}^{\rho}_{[q_1,q_2]}(m)\big) \leq e^{-\frac{\beta}{\gamma}\mathcal{A}}e^{\frac{\beta}{\gamma}\mathcal{E}_1}\left(1 + 2e^{-\frac{\beta}{\gamma}\left\{L_1\frac{\kappa(\beta,\theta)}{16}\delta\zeta_5^3\right\}}\right). \tag{4.103}$$

It remains to check that $\mathcal{E}_1 \downarrow 0$. Recalling (2.64), one has $\gamma\log(\rho/\gamma) \leq (g(\delta^*/\gamma))^{-1}$. Recalling (2.66) one has $(R_2+\ell_0)\sqrt{\gamma/\delta^*} \leq (g(\delta^*/\gamma))^{-1}$. Therefore, using (4.100), (4.94) and recalling that $0 < b < 1/(8+4a)$, see Proposition 4.2, one has

$$\mathcal{E}_1 \leq K(Q)(f(Q)+1)\big[\zeta_5 + 32\theta L_1\sqrt{\frac{\gamma}{\delta^*}} + \frac{c(\beta,\theta)}{(g(\delta^*/\gamma))^{1/(8+4a)}}\big] \leq (g(\delta^*/\gamma))^{-b}. \tag{4.104}$$

So one gets the upper bound in (4.11). Recalling (4.90), it is easy to get the corresponding lower bound. ■

Lemma 4.18 *For $\omega \in \Omega_1$ and choosing the parameters as in Subsection 2.5 we have*

$$e^{-\frac{\beta}{\gamma}(\mathcal{F}^*+32\theta R_2\sqrt{\frac{\gamma}{\delta^*}})} \le \frac{Z^{m,m}_{[\ell,\ell+2R_2]}\left(\mathcal{W}_1([\ell,\ell+2R_2],R_2,\zeta)\right)}{Z^{m^*,m^*}_{[\ell,\ell+2R_2]}\left(\mathcal{P}^{\rho}_{[\ell,\ell+2R_2]}(m^*)\right)} \le e^{-\frac{\beta}{\gamma}(\mathcal{F}^*-32\theta R_2\sqrt{\frac{\gamma}{\delta^*}})}. \tag{4.105}$$

The Lemma is proven in [16], see Lemma 7.3. The random field is estimated as in Lemma 3.3 (the rough estimates). The upper bound can be easily obtained since $\mathcal{F}^*$ is the minimum amount of energy needed to go from one phase to the other, see (2.25). More care should be taken to show the lower bound, see formula (7.34) of [16].

Next we summarize in Lemma 4.19 the estimates needed to prove Lemma 4.13. Let $\mathcal{I}$ be the interval that gives rise to an elongation. Denote by $\operatorname{sign}(\mathcal{I})=1$, if $\mathcal{I}$ gives rise to a positive elongation, $\operatorname{sign}(\mathcal{I})=-1$ in the other case.

Lemma 4.19 *Let Ω_1 be the probability space of Theorem 2.3, let $\mathcal{I}\subset\Delta_Q$ be an interval that gives rise to an elongation. Then for any interval $I\subset\gamma^{-1}\mathcal{I}$ we have*

$$\frac{Z^{0,0}_{I}\left(\eta^{\delta,\zeta_1}(\ell)=-\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)}{Z^{0,0}_{I}\left(\eta^{\delta,\zeta_1}(\ell)=\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)} = e^{-\operatorname{sign}(\mathcal{I})\frac{\beta}{\gamma}\sum_{\alpha\in\epsilon^{-1}\gamma I}\chi(\alpha)}\,\frac{Z^{0,0}_{I,0}\left(\eta^{\delta,\zeta_1}(\ell)=-\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)}{Z^{0,0}_{I,0}\left(\eta^{\delta,\zeta_1}(\ell)=\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)}. \tag{4.106}$$

On Ω_1, the last ratio satisfies: For all $I\subset\gamma^{-1}\mathcal{I}\subset\gamma^{-1}[-Q,+Q]$

$$\left|\log\frac{Z^{0,0}_{I,0}\left(\eta^{\delta,\zeta_1}(\ell)=-\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)}{Z^{0,0}_{I,0}\left(\eta^{\delta,\zeta_1}(\ell)=\operatorname{sign}(\mathcal{I}),\,\forall\ell\in I\right)}\right| \le \frac{\beta}{\gamma}\frac{1}{4c^2(\beta,\theta)g(\delta^*/\gamma)} \tag{4.107}$$

where g is the function given in Subsection 2.5 and $c(\beta,\theta)$ is some positive constant that depends only on β,θ.

The proof has been done in [16], see the proofs of Lemma 6.3 and of Proposition 4.8 there. It consists essentially in extracting the leading stochastic part and estimating the remaining term by using a classical deviation inequality for Lipschitz functions of Bernoulli random variables. The corresponding Lipschitz norms are estimated using the cluster expansion. The proof is however long and tedious.

5 Probability estimates and Proof of Theorem 2.5

In this section we prove the probability estimates needed for proving the main results stated in Section 2. The proof of Theorem 2.5 is given after Lemma 5.10. This section is rather long and we divided into several subsections. In the first by using a simple and direct application of the Donsker invariance principle in the Skorohod space, we prove that the main random contribution identified in (3.42) suitably rescaled, converges in law to a bilateral Brownian process (BBM), see (5.7).

In the second subsection we recall the construction done by Neveu-Pitman, [32], to determine the h-extrema for a bilateral Brownian motion and then we adapt it to the random walk corresponding to the previous random contribution. In Subsection 5.3 we state definitions and main properties of the maximal b–elongations with excess f introduced in [16]. In Subsection 5.4, which is the most involved, we identify them with the h–extrema of Neveu–Pitman by restricting suitably the probability space we are working on. Here b, f, and h are positive constant which will be specified. In the last section we present rough estimates on the number of renewals up to time R, needed to prove the Theorem 2.3.

5.1. Convergence to a Bilateral Brownian Motion

Let $\epsilon \equiv \epsilon(\gamma)$, $\lim_{\gamma\to 0}\epsilon(\gamma)=0$, $\frac{\epsilon}{\gamma^2}>\frac{\delta^*}{\gamma}$, so that each block of length $\frac{\epsilon}{\gamma^2}$ contains at least one block $A(x)$ (see section 2.2) ; to avoid rounding problems it is assumed that $\epsilon/\gamma\delta^* \in I\!\!N$, and that the basic initial partition $A(x)$: $x \in C_{\delta^*}(I\!\!R)$ is a refinement of the present one, see (2.65) for the actual choice of ϵ. Denote by $\{\hat{W}^\epsilon(t); t\in I\!\!R\}$ the following continuous time random walk:

$$\hat{W}^\epsilon(t) \equiv \begin{cases} V_1^\epsilon(t) = \dfrac{1}{\sqrt{c(\beta,\theta,\gamma/\delta^*)}} \displaystyle\sum_{\alpha=1}^{[\frac{t}{\epsilon}]} \chi(\alpha), \quad t\ge\epsilon\,; \\ 0, \qquad -\epsilon \le t \le \epsilon\,; \\ V_2^\epsilon(-t) = \dfrac{1}{\sqrt{c(\beta,\theta,\gamma/\delta^*)}} \displaystyle\sum_{\alpha=-[\frac{t}{\epsilon}]}^{\alpha=-1} \chi(\alpha),\ t\le -\epsilon. \end{cases} \tag{5.1}$$

Here $[x]$ is the integer part of x and $\chi(\alpha)$ was defined in (3.55) for all $\alpha \in Z\!\!\!Z$. Definition (5.1) allows to see $\hat{W}^\epsilon(\cdot)$ as a trajectory in the space of real functions on the line that are right continuous and have left limit, *i.e* in the Skorohod space $D(I\!\!R, I\!\!R)$ endowed with the Skorohod topology. To define a metric that makes it separable and complete, let us denote $\Lambda_{\rm Lip}$ the set of strictly increasing Lipschitz continous function λ mapping $I\!\!R$ onto $I\!\!R$ such that

$$\|\lambda\| = \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| < \infty. \tag{5.2}$$

For $v \in D(I\!\!R, I\!\!R)$ and $T\ge 0$, let us define

$$v^T(t) = \begin{cases} v(t\wedge T), & \text{if } t\ge 0; \\ v(t\vee(-T)), & \text{if } t<0. \end{cases} \tag{5.3}$$

Define for v and w in $D(I\!\!R, I\!\!R)$

$$d(v,w) \equiv \inf_{\lambda\in\Lambda_{\rm Lip}}\left[\|\lambda\| \vee \int_0^\infty e^{-T}\sup_{t\in I\!\!R}(1\wedge(|v^T(t)-w^T(\lambda(t))|)\,{\rm d}T\right]. \tag{5.4}$$

Note that for a given $T\in R$, the quantity $|v^T(t)-w^T(\lambda(t))|$ is constant for $t > T\vee\lambda(T)$ and for $t < (-T)\wedge(\lambda(-T))$, therefore the previous supremum over $t\in I\!\!R$ is merely over $(-T)\wedge(\lambda(-T)) \le t \le T\vee\lambda(T)$. See [7] chapter 3 or [20] chapter 3 where the case of $D[0,\infty)$ is considered with all the needed details. Let us define the bilateral Brownian motion $W = (W(t); t\in I\!\!R)$ by

$$W(t) \equiv W_t = \begin{cases} B_1(t) & t\ge 0 \\ B_2(-t) & t\le 0. \end{cases} \tag{5.5}$$

with $(B_1(t), t\ge 0)$ and $(B_2(-t), t\le 0)$ two independent standard Brownian motions. Note that $E[(W(t))^2] = |t|$ for all $t\in I\!\!R$, in particular $W(0)=0$, and when $s\le 0\le t$, $E[(W(t)-W(s))^2] = t-s$. Since $\chi(\alpha)$ depends on $\epsilon=\epsilon(\gamma)$, we need the following generalization of the Donsker Invariance Principle that can be proved following step by step the proof of Billingsley [7] pg 137.

Theorem (Invariance Principle) *Let $\epsilon\equiv\epsilon(\gamma)>0$ so that $\frac{\epsilon}{\gamma^2}>\frac{\delta^*}{\gamma}$, $\lim_{\gamma\to 0}\epsilon(\gamma)=0$. Let $\mathcal{P}^\epsilon$ be the measure induced by $\{\hat{W}^\epsilon(t), t\in I\!\!R\}$ on $D(I\!\!R, \mathcal{B}(I\!\!R))$ Then as $\gamma\downarrow 0$, $\mathcal{P}^\epsilon$ converges weakly to the Wiener measure $\mathcal{P}$, under which the coordinate mapping process $W(t)$, $t\in I\!\!R$ is a bilateral Brownian motion.*

Remark: One can wonder about the coherence between the fact that $\hat{W}^\epsilon(t) = 0$, for $-\epsilon \le t \le \epsilon$ in (5.1) and $\chi(0) \neq 0$. However we have $\chi(0) \equiv \gamma \sum_{x:\delta^* x \in \tilde{A}_{\epsilon/\gamma}(0)} X(x) 1\!\!1_{\{p(x) \le (2\gamma/\delta^*)^{1/4}\}}$.

Lemma 5.1

$$\lim_{\gamma \to 0} I\!P[\chi(0) = 0] = 1 \tag{5.6}$$

Proof: Using (3.57), one gets immediately for $c > 0$, $I\!P[|\chi(0)| \ge c] \le 2e^{-\frac{c^2}{3\epsilon V^2}}$ which implies (5.6), since $\epsilon = \epsilon(\gamma) \downarrow 0$. ■

The following result is immediate

Lemma 5.2 *Set $\eta = \pm 1$, $I = [\frac{a}{\gamma}, \frac{b}{\gamma}]$ (macro scale), a and b in $I\!R$. Then, see (3.42), we obtain*

$$\lim_{\gamma \to 0} \left[-\eta\gamma \Delta^\eta \mathcal{G}(m^{\delta^*}_{\beta,I}) \right] \overset{\text{Law}}{=} V(\beta,\theta)[W(b) - W(a)]. \tag{5.7}$$

Proof: Recalling (5.1), for $\eta = \pm 1$ and $I = [\frac{a}{\gamma}, \frac{b}{\gamma}]$ in macroscopic scale with $0 < a < b$ or $a < b < 0$ one gets the following

$$\gamma \Delta^\eta \mathcal{G}(m^{\delta^*}_{\beta,I}) = -\eta\gamma \sum_{x \in \mathcal{C}_{\delta^*}(I)} X(x) = -\eta \sum_{\alpha = [\frac{a}{\epsilon}]}^{[\frac{b}{\epsilon}]} \chi(\alpha) = -\eta \sqrt{c(\beta,\theta,\gamma/\delta^*)} \left[\hat{W}^\epsilon(b) - \hat{W}^\epsilon(a) \right]. \tag{5.8}$$

When $0 \in [a, b]$, we get

$$\gamma \Delta^\eta \mathcal{G}(m^{\delta^*}_{\beta,I}) = -\eta \sqrt{c(\beta,\theta,\gamma/\delta^*)} \left[\hat{W}^\epsilon(b) - \hat{W}^\epsilon(a) \right] - \eta\chi(0). \tag{5.9}$$

Therefore, using Lemma 5.1 to take care of the $\chi(0)$ term and (3.54) we obtain (5.7). ■

Remark: Note that $I = [\frac{a}{\gamma}, \frac{b}{\gamma}]$ corresponds to $\gamma I = [a, b]$ in the Brownian scale, according to the notation in Subsection 2.2. The (5.7) is the main reason to have introduced the notion of "Brownian" scale. In this scale the main random contribution identified in (3.42) becomes a functional of a bilateral Brownian motion.

5.2. The Neveu-Pitman construction of the h–extrema for the random walk $\{\hat{W}^\epsilon\}$

We shortly recall the Neveu-Pitman construction [32], used to determine the h–extrema for the bilateral Brownian Motion $(W_t, t \in I\!R)$. Realize it as the coordinates of the set Ω of real valued functions ω on $I\!R$ which vanishes at the origin. Denote by $(\theta_t, t \in I\!R)$, the flow of translation : $[\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)]$ and by ρ the time reversal $\rho\omega(t) = \omega(-t)$. For $h > 0$, the trajectory ω of the BBM admits an h–minimum at the origin if $W_t(\omega) \ge W_0(\omega) = 0$ for $t \in [-T_h(\rho\omega), T_h(\omega)]$ where $T_h(\omega) = \inf[t : t > 0, W_t(\omega) > h]$, and $-T_h(\rho\omega) = -\inf[t > 0 : W_{-t}(\omega) > h] \equiv \sup[t < 0, W_{-t}(\omega) > h]$. The trajectory ω of the BBM admits an h–minimum (resp. h maximum) at $t_0 \in I\!R$ if $W\, o\, \theta_{t_0}$ (resp. $-W\, o\, \theta_{t_0}$) admits an h minimum at 0.

To define the point process of h–extrema for the BBM, Neveu-Pitman consider first the one sided Brownian motion $(W_t, t \ge 0, W_0 = 0)$, *i.e* the part on the right of the origin of the BBM. Denote its running maximum by

$$M_t = \big(\max(W_s\,;\, 0 \le s \le t), t \ge 0\big) \tag{5.10}$$

and define

$$\begin{aligned}&\tau = \min(t\,;\,t \geq 0, M_t - W_t = h),\\&\beta = M_\tau,\\&\sigma = \max(s\,;\,0 \leq s \leq \tau\,,\,W_s = \beta).\end{aligned} \tag{5.11}$$

The stopping time τ is the first time that the Brownian motion achieves a drawdown of size h, see [38,39]. Its Laplace transform is given by $I\!E[\exp(-\lambda\tau)] = (\cosh(h\sqrt{2\lambda}))^{-1}$, $\lambda > 0$. This is consequence of the celebrated Lévy Theorem [27] which states that $(M_t - W_t; 0 \leq t < \infty)$ and $(|W_t|; 0 \leq t < \infty)$ have the same law. Therefore τ has the same law as the first time a reflected Brownian motion reaches h. The Laplace transform of this last one is obtained applying the optional sampling theorem to the martingale $\cosh(\sqrt{2\lambda}|W_t|)\exp(-\lambda t)$. Further Neveu and Pitman proved that (β, σ) and $\tau - \sigma$ are independent and give the corresponding Laplace transforms. In particular one has

$$I\!E[e^{-\lambda\sigma}] = (h\sqrt{2\lambda})^{-1}\tanh(h\sqrt{2\lambda}). \tag{5.12}$$

Now call $\tau_0 = \tau$, $\beta_0 = \beta$, $\sigma_0 = \sigma$ and define recursively $\tau_n, \beta_n, \sigma_n$ $(n \geq 1)$, so that $(\tau_{n+1} - \tau_n, \beta_{n+1}, \sigma_{n+1} - \tau_n)$ is the (τ, β, σ)–triplet associated to the Brownian motion $\big((-1)^{n-1}(W_{\tau_n + t} - W_{\tau_n}),\, t \geq 0\big)$. By construction, for $n \geq 1$, σ_{2n} is the time of an h-maximum and for $n \geq 0$, σ_{2n+1} is the time of a h-minimum. Note that since we have considered just the part on the right of the origin, in general σ_0 is not an h maximum. The definition only requires $W_t \leq W_{\sigma_0}$ for $t \in [0, \sigma_0)$, therefore $W_{\sigma_0} = W_{\sigma_0} - B_0$ could be smaller than h. The trajectory of the BBM on the left of the origin will determine whether σ_0 is or is not an h–maximum. From the above mentioned fact that (β, σ) and $\tau - \sigma$ are independent, it follows that the variables $\sigma_{n+1} - \sigma_n$ for $n \geq 1$ are independent with Laplace transform $(\cosh(h\sqrt{2\lambda}))^{-1}$. In this way Neveu and Pitman define a renewal process on $I\!R_+$, with a delay distribution, *i.e.* the one of σ_0, that have Laplace transform (5.12).

Since the times of h-extrema for the BBM depend only on its increments, these times should form a stationary process on $I\!R$. The above one side construction does not provide stationary on the positive real axis $I\!R^+$ since the delay distribution is not the one of the limiting distribution of the residual life as it should be, see [5] Theorem 3.1. In fact the Laplace transform of limiting distribution of the residual life is given by (2.50) which is different from (5.12).

There is a standard way to get a stationary renewal process. Pick up an $r_0 > 0$, translate the origin to $-r_0$ and repeat for $(W_{t+r_0},\, t > -r_0)$ the above construction. One gets $\sigma_0(r_0)$ and the sequence of point of h-extrema $(\sigma_n(r_0), n \geq 1)$. Let $\nu(r_0) \equiv \inf(n > 0 : \sigma_n(r_0) > 0)$ be the number of renewals up to time 0 (starting at $-r_0$). In this way, $\sigma_{\nu(r_0)}(r_0)$ is the residual life at "time" zero for the Brownian motion starting at $-r_0$. So taking $r_0 \uparrow \infty$, the distribution of $\sigma_{\nu(r_0)}(r_0)$ will converge to the one of the residual life and using [5], Theorem 3.1, one gets a stationary renewal process on $I\!R^+$. So conditionally on $\sigma_1(r_0) < 0$, define $S_i(r_0) = \sigma_{\nu(r_0)+i-1}(r_0)$ for all $i \geq 1$. Then since the event $\{\sigma_1(r_0) < 0\}$ has a probability that goes to 1 as $r_0 \uparrow \infty$, one gets, as $r_0 \uparrow \infty$, a stationary renewal process on $I\!R_+$ as well on $I\!R$. Since the Laplace transform of the inter–arrival time distribution is $(\cosh(h\sqrt{2\lambda}))^{-1}$, one gets easily that the Laplace transform of the distribution of S_1 (and also of S_0) is (2.50).

With this in mind we start the construction for the random walk $\{\hat{W}^\epsilon\}$. Denote

$$V^\epsilon(t) = \begin{cases} V_1^\epsilon(t) = \dfrac{1}{\sqrt{c(\beta,\theta,\gamma/\delta^*)}}\displaystyle\sum_{\alpha=1}^{[\frac{t}{\epsilon}]}\chi(\alpha) & t \geq \epsilon, \\ 0 \qquad 0 \leq t \leq \epsilon \end{cases} \tag{5.13}$$

and $\hat{\mathcal{F}}_t^+$, $t \geq 0$ the associated σ- algebra. Define the rescaled running maximum for $V^\epsilon(t)$, $t \geq 0$

$$\sqrt{\epsilon}\hat{M}(n) = \max_{0 \leq k \leq n} V^\epsilon(k\epsilon). \tag{5.14}$$

The $\sqrt{\epsilon}$ multiplying $\hat{M}(n)$ comes from the observation, see (3.56), that $I\!\!E\left[\left(\frac{1}{\sqrt{\epsilon}}V^\epsilon(k\epsilon)\right)^2\right] = k$. For any $h > 0$, define the $\hat{\mathcal{F}}_t^+$ stopping time

$$\hat{\tau}_0(\epsilon) \equiv \hat{\tau}_0 = \min\{n \geq 0 : \sqrt{\epsilon}\hat{M}(n) - V^\epsilon(n\epsilon) \geq h\}, \tag{5.15}$$

$$\sqrt{\epsilon}\hat{\beta}_0(\epsilon) \equiv \sqrt{\epsilon}\hat{\beta}_0 = \max\{V^\epsilon(k\epsilon) : 1 \leq k \leq \hat{\tau}_0\} \tag{5.16}$$

and

$$\hat{\sigma}_0(\epsilon) \equiv \hat{\sigma}_0 = \max\{k : 1 \leq k \leq \hat{\tau}_0; V^\epsilon(k\epsilon) = \sqrt{\epsilon}\hat{\beta}_0\}. \tag{5.17}$$

By construction

$$\sqrt{\epsilon}\hat{\beta}_0 \equiv \sqrt{\epsilon}\hat{M}(\hat{\tau}_0) = \max_{0 \leq k \leq \hat{\tau}_0} V^\epsilon(k\epsilon) = V^\epsilon(\hat{\sigma}_0\epsilon) \geq V^\epsilon(\hat{\tau}_0\epsilon) + h. \tag{5.18}$$

It follows from the invariance principle and the continuous mapping theorem, Theorem 5.2 of [7], that the joint distribution of

$$\left[\sqrt{\epsilon}\hat{M}([\frac{t}{\epsilon}]), \epsilon\hat{\tau}_0(\epsilon), \sqrt{\epsilon}\hat{\beta}_0(\epsilon), \epsilon\hat{\sigma}_0(\epsilon)\right]$$

converges as $\epsilon \to 0$, to the joint distribution of the respective quantities defined for a Brownian motion, see (5.11) *i.e*

$$\left[M_t, \tau_0, \beta_0, \sigma_0\right].$$

Since $\hat{\tau}_0$ is a $\hat{\mathcal{F}}_t^+$ stopping time for $(V^\epsilon(t), t \geq 0)$, the translated and reflected motion $(-1)[V^\epsilon(\epsilon\tau_0 + t) - V^\epsilon(\epsilon\tau_0)]$, for $t \geq 0$, is a new random walk independent of $(V^\epsilon(t), 0 \leq t \leq \epsilon\tau_0)$ on which we will iterate the previous construction. We have

$$\begin{aligned}
\hat{\tau}_1(\epsilon) \equiv \hat{\tau}_1 &= \min\{n \geq \hat{\tau}_0 : \max_{\tau_0 \leq k \leq n} [-V^\epsilon(k\epsilon)] + V^\epsilon(n\epsilon) \geq h\} \\
&= \min\{n \geq \hat{\tau}_0 : -\min_{\tau_0 \leq k \leq n} V^\epsilon(k\epsilon) + V^\epsilon(n\epsilon) \geq h\} \\
&= \min\{n \geq \hat{\tau}_0 : \min_{\tau_0 \leq k \leq n} V^\epsilon(k\epsilon) - V^\epsilon(n\epsilon) \leq -h\}
\end{aligned} \quad . \tag{5.19}$$

$$\begin{aligned}
\sqrt{\epsilon}\hat{\beta}_1(\epsilon) \equiv \sqrt{\epsilon}\hat{\beta}_1 &= \max\{(-V^\epsilon(k\epsilon)) : \hat{\tau}_0 \leq k \leq \hat{\tau}_1\} \\
&= -\min\{V^\epsilon(k\epsilon) : \hat{\tau}_0 \leq k \leq \hat{\tau}_1\}
\end{aligned} \tag{5.20}$$

$$\hat{\sigma}_1(\epsilon) \equiv \hat{\sigma}_1 = \max\{k : \hat{\tau}_0 \leq k \leq \hat{\tau}_1; -V^\epsilon(k\epsilon) = \sqrt{\epsilon}\hat{\beta}_1\} \tag{5.21}$$

Now for any $i \in I\!\!N$, we can iterate the above procedure to get as Neveu and Pitman the family

$$\left[\sqrt{\epsilon}\tilde{M}([\frac{t}{\epsilon}]), \epsilon\hat{\tau}_0(\epsilon), \sqrt{\epsilon}\hat{\beta}_0(\epsilon), \epsilon\hat{\sigma}_0(\epsilon), \ldots, \epsilon\hat{\tau}_i(\epsilon), \sqrt{\epsilon}\hat{\beta}_i(\epsilon), \epsilon\hat{\sigma}_i(\epsilon)\right]. \tag{5.22}$$

Using again the invariance principle and the continuous mapping theorem one gets that

$$\lim_{\epsilon \downarrow 0}\left[\epsilon\hat{\tau}_i(\epsilon), \sqrt{\epsilon}\hat{\beta}_i(\epsilon), \epsilon\hat{\sigma}_i(\epsilon), i \geq 0, i \in I\!\!N\right] \overset{\text{Law}}{=} \left[\tau_i, \beta_i, \sigma_i, i \geq 0, i \in I\!\!N\right], \tag{5.23}$$

where the quantity in the right hand side of (5.23) are the ones defined after (5.12). Let us note the following properties of the previous points of h–extrema. By construction the random walk satisfies, in the interval $[\hat\sigma_0, \hat\sigma_1]$, the following :

Property (4.A) In the interval $[\hat\sigma_0, \hat\sigma_1]$ we have

$$V^\epsilon(\hat\sigma_1\epsilon) - V^\epsilon(\hat\sigma_0\epsilon) \le -h, \qquad V^\epsilon(k\epsilon) - V^\epsilon(k'\epsilon) < h \qquad \forall k' < k \in [\hat\sigma_0, \hat\sigma_1], \tag{5.24}$$

$$V^\epsilon(\hat\sigma_1\epsilon) \le V^\epsilon(k\epsilon) \le V^\epsilon(\hat\sigma_0\epsilon) \qquad \hat\sigma_0 < k < \hat\sigma_1. \tag{5.25}$$

The first property in (5.24) is easily obtained. Namely adding and subtracting $V^\epsilon(\epsilon\hat\tau_0)$ one has

$$[V^\epsilon(\epsilon\hat\sigma_1) - V^\epsilon(\epsilon\hat\tau_0)] + [V^\epsilon(\epsilon\hat\tau_0) - V^\epsilon(\epsilon\hat\sigma_0)] \le -h$$

since $[V^\epsilon(\epsilon\hat\sigma_1) - V^\epsilon(\epsilon\hat\tau_0)] \le 0$ and by construction $V^\epsilon(\epsilon\hat\sigma_0) - V^\epsilon(\epsilon\hat\tau_0) \ge h$. The other properties are easily checked. Properties similar to (5.24) and (5.25) hold in the interval $[\hat\sigma_{2i}, \hat\sigma_{2i+1}]$, for $i > 0$. Namely by construction $\hat\sigma_{2i}$ is a point of h-maximum and $\hat\sigma_{2i+1}$ is a point of h-minimum. Further, since by construction $\hat\sigma_{2i-1}$ is a point of h-minimum and $\hat\sigma_{2i}$ is a point of h-maximum in the interval $[\hat\sigma_{2i-1}, \hat\sigma_{2i}]$, $i \ge 1$, we have the following:

Property (4.B) In the interval $[\hat\sigma_{2i-1}, \hat\sigma_{2i}]$, $i \ge 1$, we have

$$V^\epsilon(\hat\sigma_{2i}\epsilon) - V^\epsilon(\hat\sigma_{2i-1}\epsilon) \ge h, \qquad V^\epsilon(k\epsilon) - V^\epsilon(k'\epsilon) > -h \qquad \forall k' < k \in [\hat\sigma_{2i-1}, \hat\sigma_{2i}], \tag{5.26}$$

$$V^\epsilon(\hat\sigma_{2i-1}\epsilon) \le V^\epsilon(k\epsilon) \le V^\epsilon(\hat\sigma_{2i}\epsilon) \qquad \hat\sigma_{2i-1} < k < \hat\sigma_{2i}. \tag{5.27}$$

Following the Neveu–Pitman construction, one translates the origin of the random walk $\{V^\epsilon\}$ to $-r_0$, being r_0 positive and large enough and repeats the previous construction. To obtain the h–extrema as in Neveu-Pitman we should let first $\epsilon \to 0$, obtaining by the Donsker invariance principle that

$$V^\epsilon_{r_0}(\cdot) \equiv V^\epsilon(\cdot + r_0) \tag{5.28}$$

converges in law to the standard BM translated by $-r_0$, then $r_0 \to \infty$. However we cannot proceed in this way since to control some probability estimates we need to have ϵ small but different from zero. For the moment, the picture to have in mind is merely to take a suitable $r_0 = r_0(\gamma)$ that diverges when $\gamma \downarrow 0$. We denote by $(\hat\sigma_i(r_0) = \hat\sigma_i(\epsilon, r_0), i \ge 1, i \in I\!N)$ the points of h–extrema for $V^\epsilon_{r_0}(\cdot)$.

5.3. The maximal b elongations with excess f as defined in [16]

In this subsection we recall definitions of the maximal elongations from [16]. We extract them from the first 5 pages of Section 5 of [16], with different conventions that will be pointed out. This subsection is not completely self–contained since an involved probability estimate done in [16], see (5.37) is just recalled. However if one accepts it, the rest is self–contained. In [16], formula (5.3) we introduced the following

$$\mathcal{Y}(\alpha) \equiv \begin{cases} \sum_{\tilde\alpha \in [0,\alpha]} \chi(\tilde\alpha), & \text{if} \quad \alpha \ge 1; \\ 0 & \text{if} \quad \alpha = 0; \\ -\sum_{\tilde\alpha \in (\alpha,-1)} \chi(\tilde\alpha), & \text{if} \quad \alpha < -1 \end{cases} \qquad \alpha \in Z\!\!Z. \tag{5.29}$$

Definition 5.3 *Given $b > f$ positive real numbers, we say that an interval $[\alpha_1, \alpha_2]$ gives rise to a negative b–elongation with excess f, for $\mathcal{Y}(\alpha), \alpha \in \mathbb{Z}$ if*

$$\mathcal{Y}(\alpha_2) - \mathcal{Y}(\alpha_1) \leq -b - f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) \leq b - f, \quad \forall x < y \in [\alpha_1, \alpha_2]. \tag{5.30}$$

We say that $[\alpha_1, \alpha_2]$ gives rise to a positive b–elongation with excess f if

$$\mathcal{Y}(\alpha_2) - \mathcal{Y}(\alpha_1) \geq b + f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) \geq -b + f, \quad \forall x < y \in [\alpha_1, \alpha_2]. \tag{5.31}$$

In the first case we say that the sign of the b–elongation with excess f is $-$; in the second case, $+$.

Remark 5.4 . To decide if a given interval $[\alpha_1, \alpha_2]$ gives rise to a b–elongation with excess f depends only on the variables $\chi(\alpha)$ with $\alpha_1 \leq \alpha \leq \alpha_2$, i.e. it is a local procedure.

To our aim we need to determine the b–elongations with excess f which are "maximal", i.e the intervals of maximum length which give rise to a positive or negative b–elongations with excess f.

Definition 5.5 (The maximal b–elongations with excess f). *Given $b > f$ positive real numbers, the $\mathcal{Y}(\alpha)$, $\alpha \in \mathbb{Z}$, have maximal b–elongations with excess f if there exists an increasing sequence $\{\alpha_i^*, i \in \mathbb{Z}\}$ such that in each of the intervals $[\alpha_i^*, \alpha_{i+1}^*]$ we have either (1) or (2) below:*

(1) In the interval $[\alpha_i^*, \alpha_{i+1}^*]$ (negative maximal elongation):

$$\mathcal{Y}(\alpha_{i+1}^*) - \mathcal{Y}(\alpha_i^*) \leq -b - f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) \leq b - f, \quad \forall x < y \in [\alpha_i^*, \alpha_{i+1}^*]; \tag{5.32}$$

$$\mathcal{Y}(\alpha_{i+1}^*) \leq \mathcal{Y}(\alpha) \leq \mathcal{Y}(\alpha_i^*), \quad \alpha_i^* \leq \alpha \leq \alpha_{i+1}^*. \tag{5.33}$$

(2) In the interval $[\alpha_i^*, \alpha_{i+1}^*]$ (positive maximal elongation):

$$\mathcal{Y}(\alpha_{i+1}^*) - \mathcal{Y}(\alpha_i^*) \geq b + f; \quad \mathcal{Y}(y) - \mathcal{Y}(x) \geq -b + f, \quad \forall x < y \in [\alpha_i^*, \alpha_{i+1}^*]; \tag{5.34}$$

$$\mathcal{Y}(\alpha_i^*) \leq \mathcal{Y}(\alpha) \leq \mathcal{Y}(\alpha_{i+1}^*), \quad \alpha_i^* \leq \alpha \leq \alpha_{i+1}^*. \tag{5.35}$$

Moreover, if in the interval $[\alpha_i^*, \alpha_{i+1}^*]$ we have (5.32) and (5.33) (resp. (5.34) and (5.35)) then in the adjacent interval $[\alpha_{i+1}^*, \alpha_{i+2}^*]$ we have (5.34) and (5.35) (resp. (5.32) and (5.33)). At last, we make the convention

$$\alpha_0^* \leq 0 < \alpha_1^*. \tag{5.36}$$

Remark 5.6 . In [16] the convention $\alpha_{-1}^* \leq 0 < \alpha_0^*$ was assumed.

We say that the interval $[\alpha_i^*, \alpha_{i+1}^*]$ gives rise to negative *maximal* b elongations with excess f in the first case and the interval $[\alpha_i^*, \alpha_{i+1}^*]$ gives rise to positive *maximal* b elongations with excess f in the second case.

Remark 5.7 . Note that if $\{\alpha_i^*, i \in \mathbb{Z}\}$ gives rise to *maximal* b elongations with excess $f > 0$, then $\{\alpha_i^*, i \in \mathbb{Z}\}$ gives rise to *maximal* b elongations with excess f' with $0 \leq f' \leq f$.

The α_i^* are in fact $\alpha_i^* \equiv \alpha_i^*(\gamma, \epsilon, b, f, \omega)$. We will write explicitly the dependence on one, some or all the parameters only when needed. Since we are considering a random walk and α_i^* are points of local extrema, see (5.33) and (5.35), for a given realization of the random walk, various sequences $\{\alpha_i^*, i \in \mathbb{Z}\}$ could have the properties listed above. This because a random walk can have locally and globally multiple maximizers or minimizers. Almost surely this does not happen for the Brownian motion. In [16], we have chosen to take the first minimum time or the first maximum time instead of the last one as in (5.11). In the Brownian motion case the last and first maximum (resp. minimum) time are almost surely equal. However we could have taken the last minimum time or the last maximum time without any substantial change. From now on, we make this last choice. With this choice and the convention (5.36) the points α_i^* are unambiguously defined. The interval $[\alpha_0^*, \alpha_1^*]$ is called maximal b–elongation with excess f that contains the origin.

Remark 5.8 . Obviously the construction of maximal b–elongation with excess f cannot be a local procedure. So to determine $[\alpha_0^*, \alpha_1^*]$, for example, implies to know that the intervals adjacent to $[\alpha_0^*, \alpha_1^*]$ give risen to b–elongation with excess f (not necessarily maximal) of sign opposite to the one associated to $[\alpha_0^*, \alpha_1^*]$.

In [16] we determined the maximal b– elongation with excess f containing the origin and estimated the $I\!P$-probability of the occurrence of $[\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon]$ taking care of ambiguities mentioned above. Namely, applying 5.8, 5.9 and Corollary 5.2 of [16], choosing δ^*, Q and ϵ as in Subsection 2.5, $b = 2\mathcal{F}^*$, and see (5.30) in [16], $f = 5/g(\delta^*/\gamma)$, we have proved

$$
\begin{aligned}
&I\!P\left[([\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon])^c\right] \leq 3e^{-\frac{Q}{2C_1}} + \epsilon^{\frac{a}{16(2+a)}} + Q^2 \epsilon^{\frac{a}{8+2a}} + Q e^{-\frac{1}{2\epsilon^{3/4} V^2(\beta,\theta)}} \\
&\leq \epsilon^{\frac{a}{32(2+a)}} = \left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{8(2+a)}}.
\end{aligned} \tag{5.37}
$$

where $C_1 \equiv C_1(\beta, \theta)$ is an explicit constant, $V(\beta, \theta)$ as in (2.36) and $a > 0$. Estimate (5.37) is obtained in [16] estimating the probability to have enough b–elongation with excess f (not necessarily maximal) within $[-Q/\epsilon, Q/\epsilon]$ to be sure that there exists a maximal one containing the origin. Here we have a slightly different point of view, we want to be able to construct *all* the maximal b–elongations with excess f that are within $[-Q/\epsilon, Q/\epsilon]$. After a moment of reflection, one realizes that the simultaneous occurrence of the events that two b–elongations with excess f not necessarily maximal with opposite sign on the right of $[-Q/\epsilon, Q/\epsilon]$ and the same on its left should allow to construct all the maximal b–elongations with excess f that are within $[-Q/\epsilon, Q/\epsilon]$. There is a simple device used constantly in [16], to estimate the $I\!P$–probability of the simultaneous occurrence of such events on $[Q/\epsilon, (Q+L)/\epsilon]$ and on $[-(Q+L)/\epsilon, -Q/\epsilon]$ for some $L > 0$. Let us call these events $\Omega_L^-(Q, f, b)$ and $\Omega_L^+(Q, f, b)$. Since it is rather long to introduce this device and it will be used for other purposes, we postpone to the Subsection 5.5 the proof that choosing the parameter as in Subsection 2.5, taking $L = cte \log(Q^2 g(1/\gamma))$, one gets

$$
I\!P[\Omega_L^-([Q, f, b) \cap \Omega_L^+(Q, f, b)] \geq 1 - 2\epsilon^{\frac{a}{32(2+a)}} \tag{5.38}
$$

for some $a > 0$, see after (5.84). Let us call

$$
\Omega_L([-Q, +Q], f, b, 0) \equiv \Omega_L^-([Q, f, b) \cap \{[\alpha_0^*, \alpha_1^*] \subset [-Q/\epsilon, +Q/\epsilon]\} \cap \Omega_L^+(Q, f, b) \tag{5.39}
$$

where 0 in the argument of $\Omega_L(\cdot)$ is to recall that $\mathcal{Y}(0) = 0$. The space $\Omega_L([-Q, +Q], f, b, 0)$ depends on the variables $\chi(\alpha)$ for $\epsilon\alpha \in [-Q - L, Q + L]$. Collecting (5.37) and (5.38) one gets

$$
I\!P[\Omega_L([-Q, +Q], f, b, 0)] \geq 1 - 3\epsilon^{\frac{a}{32(2+a)}}. \tag{5.40}
$$

On $\Omega_L([-Q, Q], f, b, 0)$ we have

$$
-\frac{Q}{\epsilon} < \alpha^*_{\kappa^*(-Q)+1} \leq \ldots \leq \alpha_0^* < 0 < \alpha_1^* \leq \ldots \alpha^*_{\kappa^*(Q)-1} < \frac{Q}{\epsilon}. \tag{5.41}
$$

where $\kappa^*(\pm Q)$ are defined in (2.40). The construction done in [16], just described is a *bilateral* construction. We considered the process $\mathcal{Y}(\cdot)$, $\mathcal{Y}(0) = 0$, see (5.29) and we determined to the right and to the left of the origin the $b-$ elongations with excess f. The Neveu–Pitman construction, recalled in Subsection 5.2 is a *one* *side* construction. The determination of the points S_i is achieved moving the origin of the BM to $-r_0$, and then letting $r_0 \to \infty$. To be able to compare what we just recalled with the Neveu–Pitman construction for the random walk we translate the origin to $-r_0 = -4pQ$, for some $p \in I\!N$, and called $\mathcal{Y}_{r_0}$ the new random walk with $\mathcal{Y}_{r_0}(r_0) = 0$. We want to construct in the interval $[-Q/\epsilon, +Q/\epsilon]$ the maximal b–elongations with excess f for the process $(\mathcal{Y}_{r_0}(\alpha), \alpha \in Z\!\!Z)$ considering as above, extra elongations on the left and on the right of $[-Q/\epsilon, +Q/\epsilon]$. In this way we are able to compare in the same probability space the construction done in [16] with the one by Neveu-Pitman specialized for the random walk $\mathcal{Y}_{r_0}$.

Repeating step by step the construction of maximal b–elongations with excess f given in [16], and recalling (5.39), on a subset $\Omega_L([-Q,+Q], f, b, r_0)$ that depends only of the variables $\chi(\alpha)$ for $\epsilon\alpha \in [-Q-L, Q+L]$ (and in particular does not depends of the variables $\chi(\alpha)$ for $\epsilon\alpha \in [-Q-L-r_0, -Q-L-1]$), we can construct all the maximal elongations that are within $[-Q,+Q]$ for the process $(\mathcal{Y}_{r_0}(\alpha), \alpha \in Z\!\!Z)$. By translation invariance, using (5.40) we have

$$I\!P[\Omega_L([-Q,+Q], f, b, r_0)] = I\!P[\Omega_L([-Q,+Q], f, b, 0)] \geq 1 - 3\epsilon^{\frac{a}{32(2+a)}}. \tag{5.42}$$

Similarly to (5.41) we have on $\Omega_L([-Q,+Q], f, b, r_0)$

$$-\frac{Q}{\epsilon} < \alpha^*_{\kappa^*(-Q,r_0)}(r_0) + 1 \leq \ldots \leq \alpha^*_0(r_0) < 0 < \alpha^*_1(r_0) \leq \ldots \alpha^*_{\kappa^*(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon} \tag{5.43}$$

where

$$\kappa^*(-Q, r_0) = \sup(i \geq 1 \,:\, \epsilon\alpha^*_i(r_0) < -Q) \tag{5.44}$$

and

$$\kappa^*(Q, r_0) = \inf(i \geq \kappa^*(-Q, r_0) \,:\, \epsilon\alpha^*_i(r_0) > Q) \tag{5.45}$$

with the usual convention $\inf(\emptyset) = +\infty$.

Since the previous construction depends only on the increments of $\mathcal{Y}(\alpha)$ and is exactly the one used to construct $(\alpha^*_i, i \,:\, -Q < \epsilon\alpha^*_i < Q)$, we have

$$\begin{aligned} &(\alpha^*_i(r_0), \forall i \in Z\!\!Z \,:\, \kappa^*(-Q, r_0) < i < \kappa^*(Q, r_0)) \text{ on } \Omega_L([-Q,+Q], f, b, r_0) \\ &\overset{\text{Law}}{=} (\alpha^*_i, \forall i \in Z\!\!Z \,:\, \kappa^*(-Q) < i < \kappa^*(Q)) \text{ on } \Omega_L([-Q,Q], f, b, 0). \end{aligned} \tag{5.46}$$

Here X on $\Omega_1 =^{\text{Law}} Y$ on Ω_2 means that the respective conditional distributions are the same. Note that we have $\alpha^*_0(r_0) < 0 < \alpha^*_1(r_0)$ and $\alpha^*_0 < 0 < \alpha^*_1$. In particular (5.46) implies that $\alpha^*_1(r_0)$ on $\Omega_L([-Q,+Q], f, b, r_0)$ and α^*_1 on $\Omega_L([-Q,Q], f, b, 0)$ have the same law.

5.4. Relation between h–extrema and maximal b–elongation with excess f

Recalling (5.1), we have

$$\mathcal{Y}(\alpha) = \sqrt{c(\beta, \theta, \gamma/\delta^*)}\hat{W}^\epsilon(\alpha\epsilon), \quad \forall \alpha \in Z\!\!Z. \tag{5.47}$$

Furthermore taking into account that $(\hat{\sigma}_i(r_0), i \geq 1)$ are the times of h–extrema for the random walk $V^\epsilon_{r_0}$ starting a $-r_0 = -4pQ$, see the end of Subsection 5.2, and the properties (4.A) and (4.B) satisfied by $(\hat{\sigma}_i(r_0), i \geq 1)$ one recognizes immediately that the intervals $[\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0))$ for $i \geq 1, i \in I\!N$ give rise to maximal $b = h\sqrt{c(\beta, \theta, \gamma/\delta^*)}$ elongations with excess $f = 0$, for any $b > 0$. Let us define

$$\hat{\kappa}(-Q, r_0) = \sup\left(i \geq 1 : \epsilon\hat{\sigma}_i(r_0) < -Q\right). \tag{5.48}$$

We impose $i \geq 1$ in (5.48) so that $\hat{\sigma}_{\hat{\kappa}(-Q,r_0)}(r_0)$ is a time of a h–extremum. Recall that $\hat{\sigma}_0(r_0)$ may not be a point of h–extrema. Furthermore we define

$$\hat{\nu}(r_0) = \inf\left(i \geq \hat{\kappa}(-Q, r_0) : \epsilon\hat{\sigma}_i(r_0) > 0\right) \tag{5.49}$$

and

$$\hat{\kappa}(Q,r_0) = \inf\left(i \geq \hat{\nu}(r_0) : \epsilon\hat{\sigma}_i(r_0) > Q\right). \tag{5.50}$$

Note that on $\{\hat{\kappa}(Q,r_0) < \infty\}$, there are $\hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) + 1$ points of h–extremum within $[-Q,+Q]$. So let

$$\Omega_0(Q,r_0) \equiv \{\omega \in \Omega,\, \hat{\kappa}(-Q,r_0) < \nu(r_0) < \hat{\kappa}(Q,r_0) < \infty, \hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) \geq 1\} \tag{5.51}$$

be the set of realizations such that there exists at least one interval $[\epsilon\hat{\sigma}_i(r_0), \epsilon\hat{\sigma}_{i+1}(r_0)] \subset [-Q,Q]$, for some $i \in \mathbb{Z}$ with $\hat{\sigma}_i(r_0)$ and $\hat{\sigma}_{i+1}(r_0)$ that are h–extrema of $V^{\epsilon}_{r_0}(\cdot)$. On $\Omega_0(Q,r_0)$ we have

$$-\frac{Q}{\epsilon} < \hat{\sigma}_{\hat{\kappa}(-Q,r_0)+1}(r_0) < ... < \hat{\sigma}_{\hat{\nu}(r_0)-1}(r_0) < 0 < \hat{\sigma}_{\hat{\nu}(r_0)}(r_0) < ... < \hat{\sigma}_{\hat{\kappa}(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon}. \tag{5.52}$$

Note that $\Omega_0(Q,r_0) \supset \Omega_L([-Q,+Q],b,f,r_0)$. Namely, see Remark 5.7, if $[\epsilon\alpha^*_i(f,r_0), \epsilon\alpha^*_{i+1}(f,r_0))$ gives rise to a maximal b–elongation with excess f, then it gives rise to a maximal b–elongation with excess $f = 0$. Therefore $\epsilon\alpha^*_i(f,r_0)$ and $\epsilon\alpha^*_{i+1}(f,r_0)$ are points of $h = b/\sqrt{c(\beta,\theta,\delta^*/\gamma)}$ extrema. Of course, it could exist a pair of points of h–extrema, $h = b/\sqrt{c(\beta,\theta,\delta^*/\gamma)}$, $\epsilon\hat{\sigma}_i(r_0), \epsilon\hat{\sigma}_{i+1}(r_0)$ for $\hat{\kappa}(-Q,r_0) \leq i < \hat{\kappa}(Q,r_0)$ such that $[\epsilon\hat{\sigma}_i(r_0), \epsilon\hat{\sigma}_{i+1}(r_0))$ gives rise to a maximal b–elongation with excess $f = 0$ without giving rise to a maximal b–elongation with excess $f > 0$. That is, a priori on $\Omega_0(Q,r_0) \cap \Omega_L([-Q,+Q],b,f,r_0) = \Omega_L([-Q,+Q],b,f,r_0)$, we have $\hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) > \kappa^*(Q,r_0) - \kappa^*(-Q,r_0)$.

Lemma 5.9 *Set $b = 2\mathcal{F}^*$, $h = \frac{2\mathcal{F}^*}{\sqrt{c(\beta,\theta,\delta^*/\gamma)}}$, all the remaining parameters as in Subsection 2.5, $L = cte\log(Q^2 g(\frac{\delta^*}{\gamma}))$ and $f = \frac{5}{g(\frac{\delta^*}{\gamma})}$. Set*

$$\Omega(f,r_0) = \Omega_L([-Q,+Q],b,f,r_0) \cap \{\hat{\kappa}(Q,r_0) - \hat{\kappa}(-Q,r_0) > \kappa^*(Q,r_0) - \kappa^*(-Q,r_0)\}. \tag{5.53}$$

We have

$$\mathbb{P}[\Omega(f,r_0)] \leq 3e^{-\frac{Q}{2C_1}} + \epsilon^{\frac{a}{16(2+a)}} + Q^2\epsilon^{\frac{a}{8+2a}} + Qe^{-\frac{1}{2\epsilon^{3/4}V^2(\beta,\theta)}} \leq \epsilon^{\frac{a}{32(2+a)}}. \tag{5.54}$$

where $C_1 \equiv C_1(\beta,\theta)$ is an explicit constant, $V(\beta,\theta)$ as in (2.36) and $a > 0$.

Proof: Denote

$$\begin{aligned}\Omega' = \Big\{\omega :\, & -\frac{Q}{\epsilon} < \hat{\sigma}_{\hat{\kappa}(-Q,r_0)+1}(r_0) < \ldots < \hat{\sigma}_{\hat{\nu}(r_0)-1}(r_0) < 0 < \hat{\sigma}_{\nu(r_0)}(r_0) < \ldots < \hat{\sigma}_{\hat{\kappa}(Q,r_0)-1}(r_0) < \frac{Q}{\epsilon};\\ & \exists i,\, \hat{\kappa}(-Q,r_0)+1 \leq i \leq \hat{\kappa}(Q,r_0) - 2 \text{ such that } [\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0)) \text{ does not satisfy (1) and (2) of}\\ & \text{Definition 5.5 but does satisfy (5.24) and (5.25) or (5.26) and (5.27)} \Big\}\end{aligned} \tag{5.55}$$

Note that

$$\Omega(f,r_0) \subset \Omega' \cap \Omega_L([-Q,+Q],f,b,r_0). \tag{5.56}$$

To estimate the $\mathbb{P}$–probability of the event in the right hand side of (5.56), let i, $\hat{\kappa}(-Q,r_0)+1 \leq i \leq \hat{\kappa}(Q,r_0) - 2$ be such that $[\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0)]$ does not satisfy (1) and (2) of Definition 5.5 but does satisfy (5.24) and (5.25) or (5.26) and (5.27).

It is enough to consider the case where $[\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0)]$ does not satisfy (1) of Definition 5.5 but does satisfy (5.24) and (5.25). There are two cases:

- first case

$$-b-f \leq \mathcal{Y}(\hat{\sigma}_{i+1}(r_0)) - \mathcal{Y}(\hat{\sigma}_i(r_0)) \leq -b, \quad \mathcal{Y}(y) - \mathcal{Y}(x) \leq b - f \quad \forall x,y:\, x<y \in [\hat{\sigma}_i(r_0), \hat{\sigma}_{i+1}(r_0)] \tag{5.57}$$

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) < \mathcal{Y}(\alpha) \le \mathcal{Y}(\hat\sigma_i(r_0)) \qquad \forall \alpha:\ \hat\sigma_i(r_0) < \alpha \le \hat\sigma_{i+1}(r_0) \tag{5.58}$$

- second case

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) - \mathcal{Y}(\hat\sigma_i(r_0)) \le -b-f,\ \exists x_0, y_0,\ x_0 < y_0 \in [\hat\sigma_i(r_0), \hat\sigma_{i+1}(r_0)]:\ b \ge \mathcal{Y}(y_0) - \mathcal{Y}(x_0) \ge b-f, \tag{5.59}$$

$$\mathcal{Y}(\hat\sigma_{i+1}(r_0)) < \mathcal{Y}(\alpha) \le \mathcal{Y}(\hat\sigma_i(r_0)) \qquad \hat\sigma_i(r_0) < \alpha \le \hat\sigma_{i+1}(r_0). \tag{5.60}$$

Let us denote

$$\mathcal{Y}^*(\underline{\alpha}, \alpha_1, \alpha_2) \equiv \max_{\alpha_1 \le \tilde\alpha \le \alpha_2} \sum_{\alpha=\underline{\alpha}}^{\tilde\alpha} \chi(\alpha) \tag{5.61}$$

and

$$\mathcal{Y}_*(\underline{\alpha}, \alpha_1, \alpha_2) \equiv \min_{\alpha_1 \le \tilde\alpha \le \alpha_2} \sum_{\alpha=\underline{\alpha}}^{\tilde\alpha} \chi(\alpha) \tag{5.62}$$

where $\epsilon\underline{\alpha} = -Q$. To estimate both the cases we follow an argument already used in the proof of Theorem 5.1 in [16]. Take $\rho' = (9f)^{1/(2+a)}$, for some $a > 0$. Divide the interval $[-Q, Q]$ into blocks of length ρ' and consider the event

$$\tilde{\mathcal{D}}(Q, \rho', \epsilon) \equiv \Big\{\exists \ell, \ell',\ -Q/\rho' \le \ell < \ell' \le (Q-1)/\rho'; |\mathcal{Y}^*(\underline{\alpha}, \tfrac{\rho'\ell}{\epsilon}, \tfrac{\rho'(\ell+1)}{\epsilon}) - \mathcal{Y}_*(\underline{\alpha}, \tfrac{\rho'\ell'}{\epsilon}, \tfrac{\rho'(\ell'+1)}{\epsilon}) - b| \le 9f\Big\}.$$

Simple observations show that those ω that belong to $\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\}$ and are such that there exists i, $\hat\kappa(-Q, r_0) + 1 \le i \le \hat\kappa(Q, r_0) - 2$ such that (5.57) and (5.58) hold, belong also to $\tilde{\mathcal{D}}(Q, \rho', \epsilon)$.

For the second case, we can assume that x_0 is a local minimum and y_0 a local maximum, therefore those ω that belong to $\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\}$ and are such that there exists i, $\hat\kappa(-Q, r_0)+1 \le i \le \hat\kappa(Q, r_0)-2$ such that (5.59) and (5.60) hold, belong also to $\tilde{\mathcal{D}}(Q, \rho', \epsilon)$. Therefore we obtain that

$$\Omega' \cap \{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \le f\} \subset \tilde{\mathcal{D}}(Q, \rho', \epsilon).$$

The estimate of $I\!P\left[\tilde{\mathcal{D}}(Q, \rho', \epsilon) \cap \Omega_L([-Q, +Q], f, b, r_0)\right]$ is done in [16] where the same set $\tilde{\mathcal{D}}(Q, \rho', \epsilon)$, see pag 834 there, was considered. It is based on Lemma 5.11 and Lemma 5.12 of [16]. Here we recall the final estimate

$$\begin{aligned} & I\!P\left[\tilde{\mathcal{D}}(Q, \rho', \epsilon) \cap \Omega_L([-Q, +Q], f, b, r_0)\right] \le \\ & 8(2(Q+L)+1)^2 \frac{2\sqrt{2\pi}}{V(\beta,\theta)} (9f)^{a/(2+a)} + (2(Q+L)+1) \frac{1296}{V(\beta,\theta)} \frac{9f + (2+V(\beta,\theta))\sqrt{\epsilon \log\frac{C_1}{\epsilon}}}{(9f)^{3/(4+2a)}} \\ & + \frac{4(Q+L)}{\epsilon} e^{-\frac{f}{4\epsilon V^2(\beta,\theta)}}. \end{aligned} \tag{5.63}$$

Furthermore by Chebyshev inequality we obtain that

$$I\!P\left[\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| \ge f\}\right] \le \frac{I\!E\left[\{\max_{\alpha\in[-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)|\}\right]}{f} \le 2\left(\epsilon V_+^2 \log\{\frac{2Q}{\epsilon}\}\right)^{\frac{1}{2}} (1 + \frac{1}{\log\{\frac{2Q}{\epsilon}\}})$$

For the last inequality, see formula 5.38 in [16]. Choosing the parameters as in Subsection 2.5 we obtain the thesis. ■

On $\tilde\Omega_L([-Q,+Q],f,b,r_0)=\Omega_L([-Q,+Q],f,b,r_0)\backslash\Omega(f,r_0)$, (5.43) and (5.52) hold: a point is a beginning or an ending of an interval of maximal b–elongations with excess f if and only if it is a point of h–extremum. Relabel the variables $\hat\sigma_i(r_0)$ in (5.52) as in Neveu and Pitman, that is define

$$\hat S_i(r_0)=\hat\sigma_{\nu(r_0)+i-1}(r_0),\ \forall i\in \mathbb{Z}\ :\ \hat\kappa(-Q,r_0)\le\nu(r_0)+i-1<\hat\kappa(Q,r_0). \tag{5.64}$$

Therefore, on $\tilde\Omega_L([-Q,+Q],f,b,r_0)$, we have

$$\hat S_i(r_0)=\alpha_i^*(r_0),\ \forall i\in\mathbb{Z}\ :\ -\frac{Q}{\epsilon}\le\hat S_i(r_0)\le\frac{Q}{\epsilon}. \tag{5.65}$$

Lemma 5.10 *Take*

$$b=2\mathcal{F}^*,\qquad\qquad h=\frac{2\mathcal{F}^*}{V(\beta,\theta)},$$

all the remaining parameter as in Subsection 2.5, $L=cte\log(Q^2g(\frac{\delta^*}{\gamma}))$ *and* $f=\frac{5}{g(\frac{\delta^*}{\gamma})}$.

Let $\Omega_L([-Q,Q],f,b,0)$ *be the probability space defined in (5.39) with* $\mathbb{P}[\Omega_L([-Q,Q],f,b,0)]\ge1-3\epsilon^{\frac{a}{32(2+a)}}$ *for some* $a>0$. *Let*

$$-\frac{Q}{\epsilon}<\alpha^*_{\kappa^*(-Q)+1}<....<\alpha^*_{-1}<\alpha^*_0<0<\alpha^*_1<...<\alpha^*_{\kappa^*(Q)-1}<\frac{Q}{\epsilon}$$

be the maximal b–elongations with excess f, see (5.41), and $\{S_i,i\in\mathbb{Z}\}$ *the point process of h-extrema of the BBM defined in Neveu-Pitman [32]. We have*

$$\lim_{\gamma\to0}\epsilon(\gamma)\alpha_i^*(\epsilon(\gamma),f(\gamma))\stackrel{\text{Law}}{=}S_i\qquad i\in\mathbb{Z}. \tag{5.66}$$

Proof: This is an immediate consequence of (5.46), Lemma 5.9, (5.65), (3.54) and the continuous mapping theorem. ■

Proof of Theorem 2.5 The (2.47) is proved in Lemma 5.10. The properties of S_i are recalled in Subsection 5.2 and (2.48) is proved in [32]. To derive (2.50) let $X=S_2-S_1$ be the interarrival times of the renewal process $\{S_i,i\in\mathbb{Z}\}$. Then using

$$\int_0^\infty\lambda e^{-\lambda z}\mathbb{1}_{\{x\ge z\}}\,dz=1-e^{-\lambda x},$$

one gets

$$\mathbb{E}[e^{-\lambda S_1}]=\frac{1}{h^2\lambda}[1-\mathbb{E}[e^{-\lambda X}]]=\frac{1}{h^2\lambda}[1-\frac{1}{\cosh(h\sqrt{2\lambda})}]\qquad\text{for }\lambda\ge0.$$

The distribution (2.51) has been obtained in [26], applying the Mittag-Leffler representation for $(coshz)^{-1}$. Since $\mathbb{P}[S_1>z]=\frac{1}{h^2}\int_z^\infty\mathbb{P}[X>x]dx$, one obtains differentiating (2.51) the distribution in (2.49). ■

Proof of Corollary 2.6 Since we already proved the convergence of finite dimensional distributions see (2.47), to get (2.52) it is enough to prove that for any subsequence $\{u^*_\gamma,0<\gamma<\gamma_0\}\in BV_{\rm loc}(\mathbb{R},\{m_\beta,Tm_\beta\})$, with $\gamma \downarrow 0$, one can extract a subsequence $\{u^*_{\gamma_n}, 0 < \gamma_n < \gamma_0\}$ that convergences in Law. In fact, since $BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$ is endowed with the topology induced by the metric $d(\cdot,\cdot)$ defined in (5.4), this implies that the points of jumps of $\{u^*_{\gamma_n}, 0 < \gamma_n < \gamma_0\}$ will converge in Law to some points that by (2.47) are necessarily the $(S_i, i \in Z\!\!Z)$, this will imply (2.52).

So let $\gamma \downarrow 0$ be any subsequence that goes to 0. We will prove that for any chosen ϵ_1, it is possible to extract a subsequence $\gamma_n \downarrow 0$ and to construct a probability subset $\mathcal{K}_\epsilon \subset \Omega$ with

$$I\!P[\mathcal{K}_{\epsilon_1}] \geq 1 - \epsilon_1 \tag{5.67}$$

so that on $\mathcal{K}_{\epsilon_1}$, the subsequence $\{u^*_{\gamma_n},\, 0 < \gamma_n \leq \gamma_0\}$ is a compact subset of $BV_{\rm loc}(I\!R, \{m_\beta, Tm_\beta\})$.

To construct $\mathcal{K}_{\epsilon_1}$ and the subsequence γ_n, we use the following probability estimates. Let $b = 2\mathcal{F}^*$ and $\Omega_L([-Q,+Q], f, b, 0)$ the probability subspace defined in (5.39), $I\!P[\Omega_L([-Q,+Q], f, b, 0)] \geq 1 - 3\epsilon^{\frac{a}{32(2+a)}}$, see (5.40). On $\Omega_L([-Q,+Q], f, b, 0)$ $u^*_\gamma(\cdot)$ jumps at the points $\{\epsilon\alpha^*_i, \kappa^*(-Q)+1 \leq i \leq \kappa^*(Q)-1\}$. It was proved in Proposition 5.3 of [16] that for $i \in Z\!\!Z$ and for $0 \leq x \leq (\mathcal{F}^*)^2/(V^2(\beta,\theta) 18 \log 2)$

$$I\!P[\epsilon\alpha^*_{i+1} - \epsilon\alpha^*_i < x] \leq 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}}. \tag{5.68}$$

By Lemma 5.14, on the probability subspace $\Omega_{\rm urt}$, with $P[\Omega_{\rm urt}] \geq 1 - (\frac{5}{g(\delta^*/\gamma)})^{\frac{a}{8(2+a)}}$ for some $a > 0$, the number of jumps within $[-Q,+Q]$ is smaller than $4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q \log\left[Q^2 g(\delta^*/\gamma)\right]$. Therefore, calling

$$\Omega_Q(x,\gamma) \equiv \left\{\omega \in \Omega_{\rm urt}; \forall i : \epsilon\alpha^*_i \in [-Q,+Q], \epsilon\alpha^*_{i+1} - \epsilon\alpha^*_i > x\right\}$$

one has

$$I\!P[\Omega_Q(x,\gamma)] \geq 1 - 4(\frac{5}{g(\delta^*/\gamma)})^{\frac{a}{32(2+a)}} - \left(4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q \log\left[Q^2 g(\delta^*/\gamma)\right]\right) 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}}. \tag{5.69}$$

For any subsequence $\gamma \downarrow 0$, one can pick up a subsequence $\{\gamma_n\}$ such that

$$\sum_{n \geq 1} \left(\frac{5}{g(\delta^*(\gamma_n)/\gamma_n)}\right)^{\frac{a}{32(2+a)}} < \infty \tag{5.70}$$

and recalling that $Q = Q(\gamma) \uparrow \infty$ when $\gamma \downarrow 0$, one can take $x = x(\gamma_n) > 0$ such that

$$\sum_{n \geq 1} \left(4 + \frac{8V_+^2}{(\mathcal{F}^*)^2} Q(\gamma_n) \log\left[Q^2(\gamma_n) g(\delta^*(\gamma_n)/\gamma_n)\right]\right) 2e^{-\frac{(\mathcal{F}^*)^2}{18x(\gamma_n)V^2(\beta,\theta)}} < \infty. \tag{5.71}$$

Now using (5.69), (5.70) and (5.71), given $\epsilon_1 > 0$, one can choose $n_0 = n_0(\epsilon_1)$ such that

$$I\!P[\bigcap_{n \geq n_0} \Omega_{Q(\gamma_n)}(x(\gamma_n), \gamma_n)] \geq 1 - \epsilon_1. \tag{5.72}$$

Denote $\mathcal{K}_{\epsilon_1} \equiv \bigcap_{n \geq n_0} \Omega_{Q(\gamma_n)}(x(\gamma_n), \gamma_n)$ and we have proven (5.67).

Let $\omega \in \mathcal{K}_\epsilon$ and $\{u^*_{\gamma_n} = u^*_{\gamma_n}(\omega), n \geq n_0\}$ the above constructed subsequence. Sufficient and necessary conditions for the compactness of $\{u^*_{\gamma_n}, n \geq n_0\}$ is to exhibit for all $\tilde\epsilon$ say, $\tilde\epsilon < 1/2$ and for some numerical constant c a *finite* $c\tilde\epsilon$–net for $\{u^*_{\gamma_n}, n \geq n_0(\epsilon)\}$, see [7] pg. 217. One can also assume that $n_0 = n_0(\epsilon, \tilde\epsilon)$ is such that

$$e^{-Q(\frac{\delta^*(\gamma_{n_0})}{\gamma_{n_0}})} \leq \tilde\epsilon \tag{5.73}$$

Set $y^2 \equiv y^2_{\gamma_n} = \frac{\tilde\epsilon x(\gamma_n)}{4(1+\tilde\epsilon)}$, **let** $k_Q \in \mathbb{Z}$ and $k_{-Q} \in \mathbb{Z}$ so that $k_Q y_n^2 \le Q < (k_Q+1)y^2$ and respectively $k_{-Q}y^2 \le -Q < (k_{-Q}+1)y^2$. Denote $\mathcal{B}(y^2,Q) \subset BV_{\rm loc}$ the finite subset

$$\mathcal{B}(y^2,Q) = \begin{Bmatrix} u_0 \in BV_{\rm loc} : u_0 \quad \text{constant on} \quad [ky^2,(k+1)y^2), k \in [k_{-Q},k_Q] \cap \mathbb{Z}, \\ \forall r \ge Q, u_0(r) = u_0(k_Q); \quad \forall r \le -Q, u_0(r) = u_0(k_{-Q}) \end{Bmatrix}$$

Let $\omega \in \mathcal{K}_\epsilon$ and $k_i^* \equiv k_i^*(\omega,\gamma_n) \in \mathbb{Z}$ such that $k_i^* y^2 \le \epsilon(\gamma_n)\alpha_i^*(\omega,\gamma_n) < (k_i^*+1)y^2$, for all i such that $\epsilon\alpha^*_{i-1} \in [-Q,+Q]$. Let $u_0 \in \mathcal{B}(y^2,Q)$ such that $u_0(k_i^*y^2) = u^*_{\gamma_n}(\epsilon\alpha_i^*)$. It remains to check that $d(u^*_{\gamma_n},u_0) \le c\tilde\epsilon$ for some numerical constant c, where $d(\cdot,\cdot)$ is defined in (5.4). Let us define the following $\lambda_{\gamma_n}(.) \in \Lambda_{\rm Lip}$ by $\lambda_{\gamma_n}(k_i^*y^2) = \epsilon\alpha_i^*$ and linear between $k_i^*y^2$ and $(k_i^*+1)y^2$ for $r > Q$ take $\lambda_{\gamma_n}(r) = \lambda_{\gamma_n}(Q) + t - Q$ and for $r \le -Q$ take $\lambda_{\gamma_n}(r) = \lambda_{\gamma_n}(-Q) + t + Q$. For all i such that $\epsilon\alpha^*_{i-1} \in [-Q,+Q]$, one has

$$\left|\lambda_{\gamma_n}(k_i^*y^2) - \lambda_{\gamma_n}(k^*_{i-1}) - (k_i^* - k^*_{i-1})y^2\right| = \left|\epsilon\alpha^*_{\ell+1} - \epsilon\alpha^*_\ell - (k_i^*-k^*_{i-1})y^2\right| \le 2y^2. \tag{5.74}$$

On the other hand on $\mathcal{K}_\epsilon$ one has $\epsilon\alpha_i^* - \epsilon\alpha^*_{i-1} \ge x(\gamma_n)$ and therefore $(k_i^*-k^*_{i-1})y^2 > x(\gamma_n) - 2y^2$. Using $2y^2 \le \tilde\epsilon(x(\gamma_n) - 2y^2)$ and (5.74), one gets

$$\left|\lambda_{\gamma_n}(k_i^*y^2) - \lambda_{\gamma_n}(k^*_{i-1}) - (k_i^*-k^*_{i-1})y^2\right| \le 2y^2 \le \tilde\epsilon(x(\gamma_n)-2y^2) \le \tilde\epsilon(k_i^*y^2 - k^*_{i-1}y^2). \tag{5.75}$$

Since λ is piecewise linear one has also, for $s < t \in [k^*_{i-1}y^2, k_i^*y^2)$

$$|\lambda_{\gamma_n}(t) - \lambda_{\gamma_n}(s) - (t-s)| \le \tilde\epsilon(t-s). \tag{5.76}$$

Since λ_{γ_n} has a slope 1 outside $[-Q,+Q]$, one gets for all $s < t \in \mathbb{R}$

$$\log(1-\tilde\epsilon) \le \log\frac{\lambda_{\gamma_n}(t) - \lambda_{\gamma_n}(s)}{t-s} \le \log(1+\tilde\epsilon). \tag{5.77}$$

Therefore, recalling (5.2), (5.77) entails $\|\lambda_{\gamma_n}\| \le 4\frac{\tilde\epsilon}{3}$ and using (5.73) to control $\int_Q^\infty e^{-T}\,dT$ in (5.4) , one gets after an easy computation $d(u^*_{\gamma_n},u_0) \le 3\tilde\epsilon$. ∎

5.5. Probability estimates

We recall the already mentioned device constantly used in [16]. Lemma 5.13, stated below, gives lower and upper bound on the α_i^*, $i \in \mathbb{Z}$, in term of suitable stopping times. We set $\hat T_0 = 0$, and define, for $k \ge 1$:

$$\begin{aligned} \hat T_k &= \inf\{t > \hat T_{k-1} : |\sum_{\alpha=\hat T_{k-1}+1}^{t} \chi(\alpha)| \ge \mathcal{F}^* + \frac{f}{2}\}, \\ \hat T_{-k} &= \sup\{t < \hat T_{-(k-1)} : |\sum_{\alpha=t+1}^{\hat T_{-(k-1)}} \chi(\alpha)| \ge \mathcal{F}^* + \frac{f}{2}\}. \end{aligned} \tag{5.78}$$

Clearly, the random variables $\Delta\hat T_{k+1} := \hat T_{k+1} - \hat T_k$, $k \in \mathbb{Z}$, are independent and identically distributed. (Note that, by definition, $\Delta\hat T_1 = \hat T_1$.)

Remark: Note that $(\hat T_i, i \in \mathbb{Z})$ was called $(\tau_i, i \in \mathbb{Z})$ in [16], we change their names to avoid ambiguities with the τ defined in (5.11) and the ones defined after (5.12).

We define,

$$\tilde{S}_k = \operatorname{sgn}\Big(\sum_{j=\hat{T}_{k-1}+1}^{\hat{T}_k} \chi(j)\Big); \qquad \tilde{S}_{-k} = \operatorname{sgn}\Big(\sum_{j=\hat{T}_{-k}+1}^{\hat{T}_{-k+1}} \chi(j)\Big) \qquad \text{for} \quad k \geq 1. \tag{5.79}$$

The following lemma estimates the probability to detect at least one $b = 2\mathcal{F}^*$ elongation, with excess f, not necessarily maximal. The proof is done in [16], Lemma 5.9 there.

Lemma 5.11 *There exists an ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, all integer $k \geq 1$, and all $s > 0$ we have*

$$I\!P\left[\hat{T}_k \leq \frac{k(s+\log 2)C_1}{\epsilon}; \exists i \in \{1,\ldots,k-1\}, \tilde{S}_i = \tilde{S}_{i+1}\right] \geq \left(1-e^{-sk}\right)(1-\frac{1}{2^{k-1}}). \tag{5.80}$$

for some $C_1 = C_1(\beta,\theta)$.

To detect elongations with alternating sign, we introduce on the right of the origin

$$\begin{aligned} i_1^* &\equiv \inf\Big\{i \geq 1 : \tilde{S}_i = \tilde{S}_{i+1}\Big\} \\ i_{j+1}^* &\equiv \inf\Big\{i \geq (i_j^*+2) : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_j^*}\Big\} \qquad j \geq 1, \end{aligned} \tag{5.81}$$

and on the left

$$i_{-1}^* \equiv \begin{cases} -1 & \text{if } \tilde{S}_{-1} = \tilde{S}_1 = -\tilde{S}_{i_1^*}, \\ \sup\Big\{i \leq -2 : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_1^*}\Big\} & \text{if } \tilde{S}_{-1} \neq \tilde{S}_1 \text{ or } \tilde{S}_1 = -\tilde{S}_{i_1^*}, \end{cases}$$

$$i_{-j-1}^* \equiv \sup\Big\{i \leq i_j^* - 2 : \tilde{S}_i = \tilde{S}_{i+1} = -\tilde{S}_{i_j^*}\Big\} \qquad j \geq 1. \tag{5.82}$$

The corresponding estimates are given by the following Lemma which was proved in [16], see Lemma 5.9 there.

Lemma 5.12 *There exists an ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, all k and L positive integers, L even, (just for simplicity of writing) and all $s > 0$ we have:*

$$I\!P\left[\hat{T}_{kL-1} \leq \frac{(kL-1)(s+\log 2)C_1}{\epsilon}, \forall_{1\leq j\leq k}\; i_j^* < jL\right] \geq \left(1-e^{-s(kL-1)}\right)\left(1-\tfrac{1}{2^{L-1}}\right)\left(1-\left(\tfrac{3}{4}\right)^{L/2}\right)^{k-1} \tag{5.83}$$

and

$$\begin{aligned} &I\!P\left[\hat{T}_{-kL} \geq \frac{-kL(s+\log 2)C_1}{\epsilon},\, \hat{T}_{L-1} \leq \frac{(L-1)(s+\log 2)C_1}{\epsilon},\, i_1^* < L,\, \forall_{1\leq j\leq k}\; i_{-j}^* > -jL\right] \\ &\geq \left(1-e^{-s(kL-1)}\right)\left(1-\tfrac{1}{2^{L-1}}\right)\left(1-\left(\tfrac{3}{4}\right)^{L/2}\right)^{k}. \end{aligned} \tag{5.84}$$

where $C_1 = C_1(\beta,\theta)$ is a constant.

Applying Lemma 5.12 with $L = cte\log(Q^2 g(\frac{\delta^*}{\gamma}))$, taking the parameters as in Subsection 2.5, one gets (5.38) by a short computation. The basic fact that was used constantly in [16] even if it was not formulated in its whole generality is the following.

Lemma 5.13 *On $\Omega_L([-Q,+Q],f,b,0)$, see (5.39), we have*

$$\hat{T}_i \le \alpha^*_{i+1},\ \forall i : 1 \le i < \kappa^*(Q) \tag{5.85}$$

and

$$\alpha^*_i \le \hat{T}_{i^*_{i+1}},\ \forall i : 1 \le i < \kappa^*(Q), \tag{5.86}$$

where $\kappa^*(Q)$ is defined in (2.40).

Proof: Recall that on $\Omega_L([-Q,+Q],f,b,0)$ we have assumed that $\alpha^*_0 \le 0 < \alpha^*_1$. To prove (5.85) we start proving that $\hat{T}_1 \le \alpha^*_2$. Suppose that $\alpha^*_2 < \hat{T}_1$. Then, from (5.78), since $\alpha^*_1 < \alpha^*_2 < \hat{T}_1$ we have

$$|\mathcal{Y}(\alpha^*_1)| < \mathcal{F}^* + \frac{f}{2} \qquad \text{and} \qquad |\mathcal{Y}(\alpha^*_2)| < \mathcal{F}^* + \frac{f}{2} \tag{5.87}$$

which is a contradiction since by assumption $[\epsilon\alpha^*_1, \epsilon\alpha^*_2]$ is a maximal $2\mathcal{F}^*$ elongation with excess f, see Definition 5.5. Similar arguments apply for $i \ge 2$. Now we prove (5.86). Assume that $[\alpha^*_0, \alpha^*_1]$ gives rise to a positive elongation. The case of a negative elongation is similar. Let us check that $\alpha^*_1 \le \hat{T}_{i^*_2}$. By definition of i^*_1, i^*_2 we have that $[\hat{T}_{i^*_1-1}, \hat{T}_{i^*_1+1}]$ is within an elongation with a sign, say $\hat{S}_{i^*_1}$ and $[\hat{T}_{i^*_2-1}, \hat{T}_{i^*_2+1}]$ is within an elongation with opposite sign, $\hat{S}_{i^*_2} = -\hat{S}_{i^*_1}$. Therefore, either $\hat{S}_{i^*_1}$ or $\hat{S}_{i^*_2}$ is negative, which implies that $\alpha^*_1 \le \hat{T}_{i^*_2}$. The general case is clearly the same. ■

Given an integer $R > 0$, we denote as in (2.40) $\kappa^*(R) = \inf\{i \ge 1 : \epsilon\alpha^*_i \ge R\}$. We define the stopping time $\tilde{k}(R) = \inf\{i \ge 0 : \epsilon\hat{T}_i \ge R\}$. By definition

$$\epsilon\hat{T}_{\tilde{k}(R)-1} < R \le \epsilon\hat{T}_{\tilde{k}(R)} \tag{5.88}$$

Using (5.85), we get that

$$R \le \epsilon\hat{T}_{\tilde{k}(R)} \le \epsilon\alpha^*_{\tilde{k}(R)+1} \tag{5.89}$$

therefore

$$\kappa^*(R) \le \tilde{k}(R) + 1. \tag{5.90}$$

Lemma 5.14 *There exists Ω_{urt}, $I\!P[\Omega_{urt}] \ge 1 - \left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{8(2+a)}}$, where $a > 0$, so that for all $R > 1$*

$$\kappa^*(R) \le 1 + \tilde{k}(R) \le 2 + \frac{4V_+^2}{(\mathcal{F}^*)^2} R \log\left[R^2 g(\delta^*/\gamma)\right] \tag{5.91}$$

and

$$\epsilon\alpha^*_{\kappa^*(R)+1} \le \frac{24 C_1 V_+^2 \log 2}{(\mathcal{F}^*)^2 \log(4/3)} R \left[\log(R^2 g(\delta^*/\gamma))\right]^2, \tag{5.92}$$

where $V_+ = V(\beta,\theta)\left[1 + (\gamma/\delta^*)^{\frac{1}{5}}\right]$ and $C_1 = C_1(\beta,\theta)$ is a positive constant .

Remark: It is well known that, almost surely, $\lim_{R\uparrow\infty} \tilde{k}(R)/R = (I\!E[\hat{T}_1])^{-1}$, see [5] Proposition 4.1.4. The estimate (5.91) allows us to have an upper bound valid uniformly with respect to $R \ge 1$ with an explicit bound on the probability. This is the main reason to have a $\log[R^2 g(d^*/\gamma)]$ in the right hand side of (5.91).

Proof: We can assume that we are on $\Omega_L([-Q,+Q],f,b,0)$. Suppose first that $\tilde{k}(R) > 1$. Since (5.88), we get

$$\frac{\epsilon\hat{T}_{\tilde{k}(R)-1}}{\tilde{k}(R)-1} < \frac{R}{\tilde{k}(R)-1} \le \frac{\epsilon\hat{T}_{\tilde{k}(R)}}{\tilde{k}(R)-1}. \tag{5.93}$$

Applying Lemma 5.7 of [16], setting $b = \mathcal{F}^* + (f/2)$ and $V_+ = V(\beta,\theta)\left[1 + (\gamma/\delta^*)^{\frac{1}{5}}\right]$, we obtain that for all s with $0 < s < (\mathcal{F}^* + (f/2))^2[4(\log 2)V_+^2]^{-1}$, for all positive integer n

$$I\!P\left[\epsilon\hat{T}_n \le ns\right] \le e^{-n\frac{(\mathcal{F}^*)^2}{4sV_+^2}}. \tag{5.94}$$

Therefore

$$I\!P\left[\exists n \ge 1 : \frac{\epsilon\hat{T}_n}{n} \le s\right] \le \frac{e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}{1 - e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}. \tag{5.95}$$

Applying (5.93), we get that for $\tilde{k}(R) > 1$

$$I\!P\left[\tilde{k}(R) \le 1 + \frac{R}{s}\right] \ge \frac{1 - 2e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}{1 - e^{-\frac{(\mathcal{F}^*)^2}{4sV_+^2}}}. \tag{5.96}$$

When $\tilde{k}(R) = 0$ or $\tilde{k}(R) = 1$, (5.96) is certainly true, therefore (5.96) holds for all $\tilde{k}(R) \ge 0$. Choosing in (5.96)

$$s_0^{-1} = \frac{4V_+^2}{(\mathcal{F}^*)^2}\left[\log R^2 g(\delta^*/\gamma)\right] \tag{5.97}$$

we get

$$I\!P\left[\forall R \ge 1,\, \tilde{k}(R) \le 1 + \frac{R}{s_0}\right] \ge 1 - \sum_{R\ge 1}\frac{\frac{2}{g(\delta^*/\gamma)R^2}}{1 - \frac{2}{g(\delta^*/\gamma)R^2}} \ge 1 - \frac{3}{g(\delta^*/\gamma)}. \tag{5.98}$$

Recalling (5.90), for all $R \ge 1$,

$$\kappa^*(R) \le 1 + \tilde{k}(R) \le 2 + \frac{4V_+^2}{(\mathcal{F}^*)^2}R\left[\log R^2 g(\delta^*/\gamma)\right] \tag{5.99}$$

which is (5.91). Next we prove (5.92). Applying (5.86) and (5.90) we have

$$\epsilon\alpha^*_{k(L)+1} \le \epsilon\hat{T}_{i^*_{\tilde{k}(L)+2}}. \tag{5.100}$$

Using (5.83) with

$$\begin{aligned} L = L_0 &= 1 + 3\frac{\log(R^2 g(\delta^*/\gamma))}{\log(4/3)} \\ k = k_0 &= 2 + \frac{4V_+^2}{(\mathcal{F}^*)^2}R\log[R^2 g(\delta^*/\gamma)]. \end{aligned} \tag{5.101}$$

After an easy computation, given $R \ge 1$ with a $I\!P$–probability greater than $1 - c(\beta,\theta)\frac{\log(R^2 g(\delta^*/\gamma))}{g(\delta^*/\gamma)^{3/2}R^2}$ we have

$$\epsilon\hat{T}_{(2+k_0)L_0} \le \frac{24C_1V_+^2\log 2}{(\mathcal{F}^*)^2\log(4/3)}R\left[\log(R^2 g(\delta^*/\gamma))\right]^2, \quad \forall j : 1 \le j \le k_0,\; i_j^* < jL_0. \tag{5.102}$$

Therefore, with a $I\!P$–probability greater than

$$1 - c(\beta,\theta)\frac{\log g(\delta^*/\gamma)}{g(\delta^*/\gamma)^{3/2}} \ge 1 - \frac{1}{g(\delta^*/\gamma)} \tag{5.103}$$

for all $R \geq 1$, (5.102) holds. Using (5.99) we have, for all $R \geq 1$,

$$1+\kappa^*(R) < \tilde{k}(R)+2 \leq 3+\frac{4V_+^2}{(\mathcal{F}^*)^2} R\left[\log R^2 g(\delta^*/\gamma)\right]. \tag{5.104}$$

Therefore collecting (5.102) and (5.104) we obtain that for all $R \geq 1$

$$i^*_{\tilde{k}(R)+2} \leq (2+k_0)L_0. \tag{5.105}$$

From which using again (5.102) and recalling (5.100), we get that for all $R \geq 1$

$$\begin{aligned} \epsilon\alpha^*_{\kappa^*(R)+1} &\leq \epsilon\hat{T}_{i^*_{\tilde{k}(R)+2}} \leq \epsilon\hat{T}_{(2+k_0)L-0} \\ &\leq \frac{24C_1V_+^2\log 2}{(\mathcal{F}^*)^2\log(4/3)} R\left[\log(R^2 g(\delta^*/\gamma))\right]^2 \end{aligned} \tag{5.106}$$

which is (5.92). Denote by Ω_{urt} the intersection of $\Omega_L([-Q,+Q],f,b,0)$ with the probability subsets in (5.98) and (5.103). Recalling (5.38) and the choice of ϵ, see (2.65), we get the Lemma. ■

Proof of Theorem 2.3:

We need to estimate the Gibbs probability of the set $\mathcal{P}^{\rho}_{\delta,\gamma,\zeta,[-Q,+Q]}(u^*_\gamma(\omega))$, see (2.37). According to the definition (2.33) we need to prove that on Ω_1 the minimal distance between two points of jump of u^*_γ is larger than $8\rho+8\delta$. Define

$$\Omega_{1,1} = \left\{\omega \in \Omega_{urt} : \forall i,\ -Q \leq \epsilon\alpha^*_i \leq Q;\ \epsilon\alpha^*_{i+1}-\epsilon\alpha^*_i \geq 8\rho+8\delta\right\}. \tag{5.107}$$

where Ω_{urt} is the probability subspace that occurs in Lemma 5.14. On Ω_{urt}, see Lemma 5.14, the total number of jumps of u^*_γ within $[-Q,+Q]$ is bounded by $2K(Q)+1$ with $K(Q)$ given in (2.35). Since the points of jumps of u^*_γ are the $\epsilon\alpha^*_i, i \in \mathbb{Z}$, from Proposition 5.3 in [16] we have that for all $i \in \mathbb{Z}$, for all $0 \leq x \leq (\mathcal{F}^*)^2/(V^2(\beta,\theta)18\log 2)$

$$I\!P[\epsilon\alpha^*_{i+1}-\epsilon\alpha^*_i < x] \leq 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}}. \tag{5.108}$$

Then one gets

$$I\!P\left[\Omega_{1,1}\right] \geq 1-\left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{8(2+a)}} - 2K(Q)e^{-\frac{(\mathcal{F}^*)^2}{18(3\rho+3\delta)V^2(\beta,\theta)}}. \tag{5.109}$$

Recalling (2.64), (2.67) and (2.63) one gets

$$I\!P\left[\Omega_{1,1}\right] \geq 1-\left(\frac{5}{g(\delta^*/\gamma)}\right)^{\frac{a}{10(2+a)}}. \tag{5.110}$$

Denote by

$$\Omega_1 = \Omega_{\gamma,K(Q)} \cap \Omega_{1,1} \tag{5.111}$$

where $\Omega_{\gamma,K(Q)}$ is the probability subspace in Theorem 2.2 of [16] and $K(Q)$ is given in (2.35). From the results stated in Theorem 2.1, 2.2 and 2.4 of [16] we obtain (2.34) and (2.37).

6 Proof of Theorem 2.4 and 2.9

6.1. Proof of Theorem 2.4

Let $\{W(r), r \in I\!R\}$ be a realization of the Bilateral Brownian motion. Let $u^*(r) \equiv u^*(r, W)$, $r \in I\!R$, be the function defined in (2.43) and (2.44). As consequence that $\mathcal{P}$ a.s the number of renewals in any finite interval is finite, we have that $\mathcal{P}$ a.s $u^* \in BV_{\mathrm{loc}}$. To prove the theorem we need to show that for $v \in BV_{\mathrm{loc}}$, $\mathcal{P}$ a.s. $\Gamma(v|u^*, W) \geq 0$, the equality holding only when $v = u^*$. Let S_0 be a point of $h-$ minimum, $h = \frac{2\mathcal{F}^*}{V(\beta,\theta)}$. This, by definition, implies that in the interval $[S_0, S_1)$

$$W(S_1) - W(S_0) \geq \frac{2\mathcal{F}^*}{V(\beta,\theta)}, \qquad W(y) - W(x) > -\frac{2\mathcal{F}^*}{V(\beta,\theta)}, \qquad \forall x < y \in [S_0, S_1) \tag{6.1}$$

$$W(S_0) \leq W(x) \leq W(S_1) \qquad S_0 \leq x \leq S_1. \tag{6.2}$$

Suppose first that v differs from $u^*(W)$ only in intervals contained in $[S_0, S_1)$. Since $u^*(r) = m_\beta$, for $r \in [S_0, S_1)$, see (2.43), set $v(r) = Tm_\beta 1\!\!I_{[r_1,r_2)}$ for $[r_1, r_2) \subset [S_0, S_1)$ and $v(r) = u^*(r)$ for $r \notin [r_1, r_2)$. When the interval $[r_1, r_2)$ is strictly contained in $[S_0, S_1)$ the function v has two jumps more than u^*. Then the value of $\Gamma(v|u^*, W)$, see (2.42), is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) = 2\mathcal{F}^* + V(\beta,\theta)[W(r_2) - W(r_1)] > 0, \tag{6.3}$$

which is strictly positive using the second property in (6.1). When $[r_1, r_2) \equiv [S_0, S_1)$ then the function v has two jumps less than u^*. Namely u^* jumps in S_0 and in S_1 and u does not. The value of $\Gamma(v|u^*, W)$ in such case is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) + \Gamma_{[S_1,S_2)}(v|u^*, W) = -2\mathcal{F}^* + V(\beta,\theta)[W(S_1) - W(S_0)] \geq 0. \tag{6.4}$$

The last inequality holds since the first property in (6.1). In the case in which $[r_1, r_2) \subset [S_0, S_1)$, $r_1 = S_0$, $r_2 < S_1$ then the function v has the same number of jumps as u^*. The value of $\Gamma(v|u^*, W)$ is

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) = V(\beta,\theta)[W(r_2) - W(S_0)] \geq 0 \tag{6.5}$$

which is still positive because of (6.2). When $[r_1, r_2) \subset [S_0, S_1)$, $r_1 > S_0$, $r_2 = S_1$ then, as in the previous case, the function v has the same number of jumps as u^* and again by (6.2),

$$\Gamma(v|u^*, W) = \Gamma_{[S_0,S_1)}(v|u^*, W) + \Gamma_{[S_1,S_2)}(v|u^*, W) = V(\beta,\theta)[W(S_1) - W(r_1)] \geq 0. \tag{6.6}$$

The case when v differs from u^*, still only in $[S_0, S_1)$, but in more than one interval can be reduced to the previous cases. For simplicity, suppose that $v(r) = Tm_\beta 1\!\!I_{[r_1,r_2)\cup[r_3,r_4)}$ for $[r_1, r_2)$ and $[r_3, r_4)$ both subsets of $[S_0, S_1)$ and $v(r) = u^*(r)$ for $r \notin [r_1, r_2) \cup [r_3, r_4)$. We have that

$$\Gamma(v|u^*, W) = \Gamma(v_1|u^*, W) + \Gamma(v_2|u^*, W) \tag{6.7}$$

where $v_1(r) = Tm_\beta 1\!\!I_{[r_1,r_2)} + u^* 1\!\!I_{[r_1,r_2)^c}$ and $v_2(r) = Tm_\beta 1\!\!I_{[r_3,r_4)} + u^* 1\!\!I_{[r_3,r_4)^c}$. Equality (6.7) is an obvious consequence of the linearity of the integral and the observation that $\Gamma(u^*|u^*, W) = 0$. Each term in (6.7) can then be treated as in the previous cases. By assumption $v \in BV_{\mathrm{loc}}$ and then the number of intervals in $[S_i, S_{i+1})$ where v might differ from u^* is $\mathcal{P}$ a.s finite. The conclusion is therefore that if $v \neq u^*$ in $[S_0, S_1)$

$$\Gamma(v|u^*, W) \geq 0. \tag{6.8}$$

When v differs from u^* in $[S_1, S_2)$ and S_1 is an $h-$ maximum one repeats the previous arguments recalling that by definition in $[S_1, S_2)$

$$W(S_2) - W(S_1) \leq -h \qquad W(y) - W(x) \leq h \qquad \forall x < y \in [S_1, S_2) \tag{6.9}$$

$$W(S_2) \leq W(x) < W(S_1) \qquad S_1 \leq x < S_2 \tag{6.10}$$

and $u^*(r) = Tm_\beta$, for $r \in [S_1, S_2)$, see (2.44). Then repeating step by step the previous scheme one concludes that $\mathcal{P}$ a.s.

$$\Gamma(v|u^*, W) \geq 0.$$

In the general case

$$\Gamma(v|u^*, W) = \sum_{i \in \mathbb{Z}} \Gamma_{[S_i, S_{i+1})}(v|u^*, W) \geq 0. \tag{6.11}$$

To prove that u^* is $I\!\!P$ a.s. the unique minimizer of $\Gamma(\cdot|u^*, W)$ it is enough to show that each term among (6.4), (6.5) and (6.6) is strictly positive, so that we get a strict inequality in (6.11). Since, see [34], page 108, exercise (3.26),

$$\mathcal{P}[\exists r \in [S_0, S_1] : [W(r) - W(S_1)] = 0] = 0,$$

we obtain that (6.6), (6.4) and by a simple argument (6.5) are strictly positive. ■

6.2. Proof of Theorem 2.9 The proof of (2.57) is an immediate consequence of Proposition 4.2 and Theorem 2.5. ■

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